

Mechanics & Fluid Dynamics

Previous year Questions from 2020 to 1992

2021-22

2020

1. Prove that the moment of inertia of a triangular lamina ABC about any axis through in A its plane is $\frac{M}{6}(\beta^2 + \beta\gamma + \gamma^2)$ where M is the mass of the lamina and β, γ are respectively the length of perpendiculars from B and C on the axis. **[10 Marks]**
2. By writing down the Hamiltonian, find the equations of motion of a particle of mass m constrained to move on the surface of a cylinder defined by $x^2 + y^2 = R^2$ is a constant. The particle is subject to a force directed towards the origin and proportional to the distance r of the particle from the origin given by $\vec{F} = -k\vec{r}$, k is a constant **[20 Marks]**
3. A velocity potential in a two-dimensional fluid flow is given by $\phi(x, y) = xy + x^2 - y^2$ find the stream function for this flow. **[15 Marks]**
4. One end of a tightly stretched flexible thin string of length l is fixed at the origin and the other at $x = l$. It is plucked at $x = \frac{1}{3}$ so that it assumes initially the shape of a triangle of height h in the $x - y$ plane. Find the displacement y at any distance x and at any time t after the string is released from rest. Take, $\frac{\text{horizontal tension}}{\text{mass per unit length}} = c^2$. **[20 Marks]**
5. Two sources of strength $\frac{m}{2}$ are placed at the points $(\pm a, 0)$. Show that at any point on the circle $x^2 + y^2 = a^2$ the velocity is parallel to the y axis and is inversely proportional to y . **[15 Marks]**

2019

6. A uniform rod OA of length $2a$ free to turn about it end O revolves with angular velocity about the vertical OZ through O and is inclined at a constant angle α to OZ ; find the value of α **[10 Marks]**
7. A circular cylinder of radius and radius a of gyration k rolls without slipping inside a fixed hollow cylinder of radius b . Show that the plane thorough axes move in a circular pendulum of length $(b - a)\left(1 + \frac{k^2}{a^2}\right)$. **[20 Marks]**
8. Using Hamilton's equation find the acceleration for a sphere rolling down a rough inclined plane, if x be the distance of the point of contact of the sphere from a fixed point on the plane. **[15 Marks]**
9. A sphere of radius R whose centre is at rest, vibrates radially in an infinite incompressible fluid of density ρ , which is at rest at infinity, if the pressure at infinity is Π , so that the pressure at the surface of the sphere at time t is $\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}$. **[15 Marks]**
10. Two sources, each of strength are places at the points $(-a, 0), (a, 0)$ and a sink of strength at origin. Show that the stream lines are the curves $(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$ where λ is a variable parameter. Show also that the fluid speed at any point is $(2a^2)/(r_1 r_2 r_3)$ where $r_1 r_2$ and r_1 re the distances of the points from the source and sings respectively. **[20 Marks]**

2018

11. For an incompressible fluid flow, two components of velocity (u, v, w) are given by $u = x^2 + 2y^2 + 3z^2$
 $v = x^2y - y^2z + zx$. Determine the third component w so that they satisfy the equation of continuity. Also,
 find the z-component of acceleration. **[10 Marks]**
12. Suppose the Lagrangian of a mechanical system is given by
 $L = \frac{1}{2}m(ax^2 + 2bxy + cy^2) - \frac{1}{2}k(ax^2 + 2bxy + cy^2)$, Where $a, b, c, m(> 0), k(> 0)$ are constants and
 $b^2 \neq ac$. write down the Lagrangian equations of motion and identify the system. **[20 Marks]**
13. The Hamiltonian of a mechanical system is given by, $H = p_1q_1 - aq_1^2 + bq_2^2 - p_2q_2$ Where a, b are the
 constants. Solve the Hamiltonian equations and show that $\frac{p_2 - bq_2}{q_1} = \text{constant}$. **[20 Marks]**
14. For a two-dimensional potential flow, the velocity potential is given by $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$.
 Determine the velocity components along the directions x and y . Also, determine the stream function ψ
 and check whether ϕ represents a possible case of flow or not. **[15 Marks]**
15. A thin annulus occupies the region. $0 \leq a \leq r \leq b, 0 \leq \theta \leq 2\pi$ The faces are insulated, Along the inner edge
 the temperature is maintained 0^0 , while along the outer edge the temperature is held at $T = K \cos \frac{\theta}{2}$
 where K is a constant. Determine the temperature distribution in the annulus. **[20 Marks]**

2017

16. Show that the moment of inertia of an elliptic area of mass M and semi-axis a and b about a semi-diameter
 of length r is $\frac{1}{4}M \frac{a^2b^2}{r^2}$. Further, prove that the moment of inertia about a tangent is $\frac{5M}{4}p^2$ where p is the
 perpendicular distance from the centre of the ellipse to the tangent. **[10 Marks]**
17. Two uniform rods AB, AC each of mass m and length $2a$, are smoothly hinged together at A and move on
 a horizontal plane. At time t , the mass centre of the rod is at the point (ξ, η) referred to fixed perpendicular
 axes Ox, Oy in the plane, and the rods make angles $\theta \pm \phi$ with Ox . Prove that the kinetic energy of the system
 is $m \left[\xi^2 + \eta^2 + \left(\frac{1}{3} + \sin^2 \phi \right) a^2 \theta^2 + \left(\frac{1}{3} + \cos^2 \phi \right) a^2 \phi^2 \right]$. Also derive Lagrange's equation of motion for the
 system if an external force with components $[X, Y]$ along the axes acts at A . **[20 Marks]**
18. A stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d . If
 V and v be the corresponding velocities of the stream and if the motion is assumed to steady and diverging
 from the vertex of the cone, then prove that $\frac{v}{V} = \frac{D^2}{d^2} e^{(u^2 - v^2)/2K}$, where K is the pressure divided by the
 density and is constant. **[15 Marks]**
19. If the velocity of an incompressible fluid at the point (x, y, z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2 - r^2}{r^5} \right)$, $r^2 = x^2 + y^2 + z^2$,
 then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$. Further, determine the
 streamlines. **[15 Marks]**

2016

20. Does a fluid with velocity $\vec{q} = \left[z - \frac{2x}{r}, 2y - 3z - \frac{2y}{r}, x - 3y - \frac{2z}{r} \right]$ possess vorticity, where $\vec{q}(u, v, w)$ is the velocity in the Cartesian frame $\vec{r}(x, y, z)$ and $r^2 = x^2 + y^2 + z^2$? What is the circulation in the circle $x^2 + y^2 = 9, z = 0$? [10 marks]
21. Consider single free particle of mass m , moving in space under no forces. If the particle starts from the origin $t = 0$ and reaches the position (x, y, z) at time τ , find the Hamilton's characteristic function S as a function of x, y, z, τ . [10 marks]
22. A simple source of strength m is fixed at the origin O in a uniform stream of incompressible fluid moving with velocity $U\vec{i}$. Show that the velocity potential ϕ at any point P of the stream is $\frac{m}{r} - Ur \cos \theta$, where $OP = r$ and θ is the angle which \vec{OP} makes with the direction \vec{i} . Find the differential equation of the streamlines and show that they lie on the surfaces $Ur^2 \sin^2 \theta - 2m \cos \theta = \text{constant}$. [15 marks]
23. The space between two concentric spherical shells of radii a, b ($a < b$) is filled with a liquid of density ρ . If the shells are set in motion, the inner one with velocity U in the x -direction and the outer one with velocity V in the y -direction, then show that the initial motion of the liquid is given by velocity potential
$$\phi = \frac{\left\{ a^3 U \left(1 + \frac{1}{2} b^3 r^{-3} \right) x - b^3 V \left(1 + \frac{1}{2} a^3 r^{-3} \right) y \right\}}{(b^3 - a^3)},$$
 where $r^2 = x^2 + y^2 + z^2$, the coordinate being rectangular. Evaluate the velocity at any point of the liquid. [20 marks]
24. A hoop with radius r is rolling, without slipping, down an inclined plane of length l and with angle of inclination ϕ . Assign appropriate generalized coordinate to the system. Determine the constraints, if any. Write down the Lagrangian equation for the system. Hence or otherwise determine the velocity of the hoop at the bottom of the inclined plane. [15 marks]

2015

25. Consider a uniform flow U_0 in the positive x -direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points. [10 Marks]
26. Calculate the moment of inertia of a solid uniform hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ with mass m about the OZ -axis. [10 Marks]
27. Solve the plane pendulum problem using the Hamiltonian approach and show that H is a constant of motion. [15 Marks]
28. A Hamiltonian of a system with one degree of freedom has form
$$H = \frac{p^2}{2\alpha} - b a p e^{-at} + \frac{b\alpha}{2} q^2 e^{-at} (\alpha + b e^{-at}) + \frac{k}{2} q^2$$
 where α, b, k are constant, q is the generalized coordinate and p is the corresponding generalized momentum.
- (i) Find a Lagrangian corresponding to this Hamiltonian.
- (ii) Find an equivalent Lagrangian that is not explicitly dependent on time. [10+10=20 Marks]
29. In an axis symmetric motion, show that stream function exists due to equation of continuity. Express the velocity components of the stream function. Find the equation satisfied by the stream function if the flow is irrotational. [20 Marks]

2014

30. Find the equation of motion of a compound pendulum using Hamilton's equations. [10 Marks]
31. Given the velocity potential $\phi = \frac{1}{2} \log \left[\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \right]$, determine the streamlines. [20 Marks]
32. Find Navier-Stokes equation for steady laminar flow of a viscous incompressible fluid between two infinite parallel plates. [20 Marks]

2013

33. Prove that the necessary and sufficient conditions that the vortex lines may be at right angles to the stream lines are $u, v, w = \mu \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$ where μ and ϕ are function of x, y, z, t . [10 Marks]
34. For solid sphere A, B, C and D, each of mass m and radius a , are placed with their centers on the four corners of a square of side b . Calculate the moment of inertia of the system about a diagonal of the square. [10 Marks]
35. Two equal rod AB and BC, each of length l , smoothly jointed at B, are suspended from A and oscillate in a vertical plane through A. Show that that the periods of normal oscillation are $\frac{2\pi}{n}$ where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}} \right) \frac{g}{l}$ [15 Marks]
36. If fluid fills the region of space on the positive side of the x -axis, which is a right boundary and if there be a sources m at the point $(0, a)$ and an equal sink at $(0, b)$ and if the pressure on the negative side be the same as the pressure at infinity, show that the resultant pressure on the boundary is $\frac{\pi \rho m^2 (a-b)^2}{\{2ab(a+b)\}}$ where ρ is the density of the fluid. [15 Marks]
37. If n rectilinear vortices of the same strength K are symmetrically arranged as generators of a circle cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)K}$. Find the velocity at any point of the liquid. [20 Marks]

2012

38. Obtain the equations governing the motion of a spherical pendulum. [12 Marks]
39. A rigid sphere of radius a is placed in a stream of fluid whose velocity in the undisturbed state is V . Determine the velocity of the fluid at any point of the disturbed stream. [12 Marks]
40. A pendulum consists of a rod of length $2a$ and mass m ; to one end of which a spherical bob of radius $\frac{a}{3}$ and mass $15m$ is attached. Find the moment of inertia of the pendulum:
(i) About an axis through the other end of the rod and at right angle to the rod. [15 Marks]
(ii) About a parallel axis through the centre of mass of the pendulum. [Given: the centre of mass of the pendulum is $\frac{a}{12}$ above the centre of the sphere]. [15 Marks]
41. Show that $\phi = xf(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q} \rightarrow 0$ as $r \rightarrow \infty$, find the surfaces of constant speed. [30 Marks]

2011

42. Let a be the radius of the base of a right circular cone of height h and mass M . Find the moment of inertia of that right circular cone about a line through the vertex perpendicular to the axis. [12 Marks]
43. The ends of a heavy rod of length $2a$ are rigidly attached to two light rings which can respectively slide on the thin smooth fixed horizontal and vertical wires O_x and O_y . The rod starts at an angle α to the horizontal with an angular velocity $\sqrt{3g(1-\sin\alpha)/2a}$ and moved downwards. Show that it will strike the horizontal wire at the end of time $-2\sqrt{a/(3g)} \log \left[\tan \left(\frac{\pi}{8} - \frac{\alpha}{4} \right) \cot \frac{\pi}{8} \right]$. [30 Marks]
44. An infinite row of the equidistance rectilinear vortices are at distance a apart. The vortices are of the same numerical strength K but they are alternately of opposite signs. Find the Complex function that determines the velocity potential and the stream function. [30 Marks]

2010

45. A uniform lamina is bounded by a parabolic arc of latus rectum $4a$ and a double ordinate at a distance b from the vertex. If $b = \frac{a}{3}(7+4\sqrt{7})$, show that two of the principal axes at the end of a latus rectum are the tangent and normal there. [12 Marks]
46. In an incompressible fluid the vorticity at every point is constant in magnitude and direction; show that the components of velocity u, v, w are solution of Laplace's equation. [12 Marks]
47. A sphere of radius a and mass m rolls down a rough plane inclined at an angle α to the horizontal. If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equation. [30 Marks]
48. When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distance from its axis, show that the path of each vortex is given by the equation $(r^2 \sin^2 \theta - b^2)(r^2 - a^2)^2 = 4a^2 b^2 r^2 \sin^2 \theta$, θ being measured from the line through the centre perpendicular to the joint of the vortices. [30 Marks]

2009

49. The flat surface of a hemisphere of radius r is cemented to one flat surface of a cylinder of the same radius and of the same material. If the length of the cylinder be l and the total mass be m , show that the moment of inertia of the combination about the axis of the cylinder is given by: $mr^2 \frac{\left(\frac{l}{2} + \frac{4}{15}r\right)}{\left(l + \frac{2r}{3}\right)}$. [12 Marks]
50. Two sources, each of strength m are placed at the point $(-a, 0), (a, 0)$ and a sink of strength $2m$ is at the origin. Show that the stream lines are the curves: $(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$ where λ is a variable parameter. Show also that the fluid speed at any point is $(2ma^2)(r_1 r_2 r_3)$ where r_1, r_2 and r_3 are the distance of the points from the source and the sink. [12 Marks]
51. A perfectly rough sphere of mass m and radius b , rests on the lowest point of a fixed spherical cavity of radius a . To the highest point of the movable sphere is attached a particle of mass m' and the system is

disturbed. Show that the oscillations are the same as of a simple pendulum of length $(a-b) \frac{4m' + \frac{7}{5}m}{m + m' \left(2 - \frac{a}{b}\right)}$.

[30 Marks]

52. An infinite mass of fluid is acted on by a force $\frac{\mu}{r^{3/2}}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere $r = C$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{\frac{1}{2}} \cdot C^{\frac{5}{4}}$.

[30 Marks]

2008

53. A circular board is placed on a smooth horizontal plane and a boy runs round the edge of it at a uniform rate. What is the motion of the centre of the board? Explain. What happens if the mass of the board and boy are equal? [12 Marks]
54. If the velocity potential of a fluid is $\phi = \frac{1}{r^3} z \tan^{-1}\left(\frac{y}{z}\right)$, $r^2 = x^2 + y^2 + z^2$ then show that the stream lines lie on the surfaces $x^2 + y^2 + z^2 = c(x^2 + y^2)^{2/3}$, c being a constant. [12 Marks]
55. A uniform rod of mass $3m$ and length $2l$ has its middle point fixed and a mass m is attached to one of its extremity. The rod, when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity $\sqrt{\frac{28}{l}}$. Show that the heavy end of the rod will fall till the inclination of the rod to the vertical is $\cos^{-1}(\sqrt{2}-1)$. [30 Marks]
56. Let the fluid fills the region $x \geq 0$ (right half of $2d$ plane). Let a source α be $(0, y_1)$ and equal sink at $(0, y_2)$, $y_1 > y_2$. Let the pressure be same as pressure infinity i.e., p_0 . Show that the resultant pressure on the boundary (y -axis) is $\pi\rho\alpha^2(y_1 - y_2)^2 / 2y_1y_2(y_1 + y_2)$, ρ being the density of the fluid. [30 Marks]

2007

57. Consider a system with two degree of freedom for which $V = q_1^2 + 3q_1q_2 + 4q_2^2$. Find the equilibrium position and determine if the equilibrium is stable. [12 Marks]
58. Show that $\left(\frac{x^2}{a^2}\right) \cos^2 t + \left(\frac{y^2}{b^2}\right) \sec^2 t = 1$ is a possible form for the boundary surface of a liquid. [12 Marks]
59. A point mass m is placed on a frictionless plane that is tangent to the Earth's surface. Determine Hamilton's equations by taking x or θ as the generalized coordinate. [30 Marks]
60. A thin plate of very large area is placed in a gap of height h with oils of viscosities μ' and μ'' on the two sides of the plate. The plate is pulled at a constant velocity V . Calculate the position of the plate so that
 (i) The shear force on the sides of the two sides of the plate is equal
 (ii) The force required to drag the plate is minimum. [End effects are neglected] [30 Marks]

2006

61. Given points $A(0,0)$ and $B(x_0, y_0)$ not in the same vertical, it is required to find a curve in the $x-y$ plane joining A to B so that a particle starting from rest will traverse from A to B along this curve without friction in the shortest possible time. If $y = y(x)$ is the required curve find the function $f(x, y, z)$ such that equation of motion can be written as $\frac{dx}{dt} = f(x, y(x), y'(x))$. [12 Marks]
62. A steady Inviscid incompressible flow has a velocity field $u = fx, v = -fy, w = 0$, where f is a constant. Derive an expression for the pressure field $p(x, y, z)$, if the pressure $p(0,0,0) = p_0$ and $F = -giz$. [12 Marks]
63. A particle of mass m is constrained to move on the surface of a cylinder. The particle is subject to a force directed towards the origin and proportional to the distance to of particle from the origin. Construct the Hamiltonian and Hamilton's equations of motion. [30 Marks]
64. Liquid is contained between two parallel planes; the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated; prove that if p be the pressure at the outer surface, the initial pressure at any point on the liquid distant r from the centre is $p \frac{\log r - \log b}{\log a - \log b}$. [30 Marks]

2005

65. A rectangular plate swings in a vertical plane about one of its corners. If its period is one second, find the length of its diagonal. [12 Marks]
66. Prove that the necessary and sufficient condition for vortex lines and stream to be at right angles to each other is that $u = \mu \frac{\partial \phi}{\partial x}, v = \mu \frac{\partial \phi}{\partial y}, w = \mu \frac{\partial \phi}{\partial z}$ where μ and ϕ are functions of x, y, z and t . [12 Marks]
67. A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M' starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time $\sqrt{\frac{2M'a}{(M+M')g \sin \alpha}}$ Where a is the length of the plank? [30 Marks]
68. State the conditions under which Euler's equation of motion can be integrated show that $-\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V \int \frac{dp}{\rho} = F(t)$ where the symbols have their usual meaning. [30 Marks]

2004

69. A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$ which is assumed frictionless. Obtain the equation of motion show that it will describe a horizontal circle in the plane $z = h$, provided that it is given an angular velocity whose magnitude is $\omega = \sqrt{\frac{2g}{a}}$. [12 Marks]
70. In an incompressible fluid the vorticity at every point is constant in magnitude and direction Do the velocity components? Justify. [12 Marks]
71. Derive the Hamilton equations of motion from the principle of least action and obtain the same for a particle of mass m moving in a force field of potential V . Write these equations in spherical coordinates (r, θ, ϕ) . [30 Marks]

72. The space between two infinitely long coaxial cylinder of radii a and b ($b > a$) respectively is filled by a homogeneous fluid of density ρ . The inner cylinder is suddenly moved with velocity v perpendicular to this axis, the outer being kept fixed. Show that the resulting impulsive pressure on a length l of inner cylinder is $\pi\rho a^2 l \frac{b^2 + a^2}{b^2 - a^2} v$. [30 Marks]

2003

73. A solid body of density ρ is in the shape of the solid formed by the revolution of the Cardioid $r = a(1 + \cos \theta)$ about the initial line. Show that the moment of inertia about the straight line through the pole and perpendicular to the initial line is $\frac{352}{105} \pi \rho a^2$. [12 Marks]
74. For an incompressible homogeneous fluid at the point (x, y, z) the velocity distribution is given by $u = -\frac{c^2 y}{r^2}$, $v = \frac{c^2 x}{r^2}$, $w = 0$; where r denotes the distance from z-axis. Show that it is a possible motion and determine the surface which is orthogonal to stream line. [12 Marks]
75. A fine circular tube, radius c , lies on a smooth horizontal plane and contains two equal particles connected by an elastic string in the tube, the natural length of which is equal to half the circumference. The particles are in contact and fastened together, the string being stretched the tube. If the particles become disunited, prove that the velocity to the tube when the string has regained its natural length is $\left\{ \frac{2\pi\lambda mc}{M(M+2m)} \right\}^{\frac{1}{2}}$ where M, m the masses of the tube and each particle respectively, and is λ the modulus of elasticity. [30 Marks]
76. Two sources, each of strength m are placed at the points $(-a, 0)$ and $(a, 0)$ and a sink of strength $2m$ is placed at the origin. Show that the stream lines are the curves $(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$, where λ is a variable parameter. Also show that fluid speed at any point is $\frac{2ma^2}{r_1 r_2 r_3}$ where r_1, r_2 and r_3 are respectively the distance of the point from the sources and sink. [15 Marks]
77. An infinite mass of fluid is acted upon by a force μr^{-3} per unit mass directed the origin. If initially the fluid is at a rest and there is a cavity in the form of a sphere $r = c$ in it. Show that the cavity will be filled up after an interval of time $\left\{ \frac{2}{5\mu} \right\}^{\frac{1}{2}} \frac{5}{c^4}$. [15 Marks]

2002

78. Find the moment of inertia of a circular wire about (i) a diameter and (ii) a line through the centre and perpendicular to its plane. [12 Marks]
79. Show that the velocity potential $\phi = \frac{1}{2} a(x^2 + y^2 - 2z^2)$ satisfies the Laplace equation, and determine the stream lines. [12 Marks]
80. A thin circular disc of mass M and radius a can turn freely about a thin axis OA which is perpendicular to its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity w about its end A . Show that the inclination ϕ to the vertical of the radius of the disc through O is $\cos^{-1} \left(\frac{g}{aw^2} \right)$ unless $w^2 < \frac{g}{a}$ and then θ is zero. [30 Marks]

81. Show that: $u = \frac{-2xyz}{(x^2 + y^2)^2}$, $v = \frac{(x^2 - y^2)z}{(x^2 + y^2)^2}$, $w = \frac{y}{x^2 + y^2}$ are the velocity components of a possible liquid motion. Is this motion irrotational? [15 Marks]

82. Prove that $\left(\nu \nabla^2 - \frac{\partial}{\partial t} \right) \nabla^2 \psi = \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)}$ where ν is the kinematic viscosity of the fluid and ψ is the stream function for a two-dimensional motion of a viscous fluid. [15 Marks]

2001

83. Determine the moment of inertia of a uniform hemisphere about its axis of symmetry and about an axis perpendicular to the axis of symmetry and through centre of the base. [12 Marks]

84. If the velocity distribution of an incompressible fluid at the point (x, y, z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{kz^2 - r^2}{r^5} \right)$ then determine the parameter k such that it is a possible motion. Hence find its velocity potential. [12 Marks]

85. Find the equation of motion for a particle of mass m which is constrained to move on the surface of a cone of semi-vertical angle α and which is subjected to a gravitational force. [30 Marks]

86. Show that the velocity distribution in axial flow of viscous incompressible fluid along a pipe of annular cross-

section radii $r_1 < r_2$, is given by $\omega(r) = \frac{1}{4\mu} \frac{dp}{dz} \left\{ r^2 - r_1^2 + \frac{r_2^2 - r_1^2}{\log \left(\frac{r_2}{r_1} \right)} \log \left(\frac{r}{r_2} \right) \right\}$. [30 Marks]

2000

87. Find the moment of inertia of an elliptic area about a line CP inclined at θ to the major axis and about a tangent parallel to CP where C is the centre of the ellipse. [12 Marks]

88. Determine the stream line and the path lines of the particle when the components of the velocity field are given by $u = \frac{x}{1+t}$, $v = \frac{y}{2+t}$ and $w = \frac{z}{3+t}$. Also state the condition for which the stream lines are identical with the path lines. [12 Marks]

89. A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M' starting from the upper end walks down the plank so that it does not move. show that he gets to the other end in time $\sqrt{\frac{2M'a}{(M+M')g \sin \alpha}}$ where a is the length of the plank. [30 Marks]

90. Define irrotational and rotational flows giving an example for each show that $u = \frac{-2xyz}{(x^2 + y^2)^2}$, $v = \frac{(x^2 - y^2)z}{(x^2 + y^2)^2}$, $w = \frac{y}{(x^2 + y^2)}$ are the velocity component of a possible liquid motion. Examine this for irrotational motion [30 Marks]

1999

91. A particle of given mass m moves in space with the Lagrangian $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V + \dot{x}A + \dot{y}B + \dot{z}C$ where V, A, B, C are given function of x, y, z . show that the equations of motion are $m\ddot{x} = -\frac{\partial V}{\partial x} + \dot{y}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) - \dot{z}\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)$ and two similar equations for y and z . Find also the Hamiltonian H in terms of generalized momenta. **[20 Marks]**
92. A wheel consists of a thin rim of mass M and n evenly placed spokes each of mass m , which may be considered as thin rods terminating at the centre of the wheel. If the wheel is rolling with linear velocity v , express its kinetic energy in terms of M, m, n, v . With what acceleration will it roll down a rough inclined plane of inclination α ? **[20 Marks]**
93. Find the moment of inertia of a solid hemisphere about a diameter of its plane base. A solid hemisphere is held with its base against a smooth vertical wall and its lowest point on the smooth floor. The hemisphere is released. Find the initial reactions on the wall and the floor. **[20 Marks]**
94. Drive the equation of continuity for a fluid in which there are no sources or sinks. Liquid flow through a pipe whose surface is the surface of revolution of the curve $y = a + kx^2/a$ about the x -axis ($-a \leq x \leq a$). If the liquid enters at the end $x = -a$ of the pipe with velocity V show that the time take by a liquid particle to traverse the entire length of the pipe from $x = -a$ to $x = +a$ is $\left\{ \frac{2a}{V(1+k)^2} \right\} \left(1 + \frac{2}{3}k + \frac{1}{5}k^2 \right)$ (Assume that k is so small that the flow remain appreciably one-dimensional throughout.) **[20 Marks]**
95. A spherical globule of gas initially of radius R_0 and at pressure P_0 expands in an infinite mass of water of density ρ in which the pressure at infinity is zero. The gas is initially at rest and its pressure p and volume v are governed by the equation $pv^{4/3} = \text{constant}$. Prove that the gas double its radius i time $\frac{28R_0}{15} \left(\frac{2\rho}{p_0} \right)^{1/2}$. **[20 Marks]**
96. Two sources, each of strength m are placed at the points $(-a, 0)$ and $(a, 0)$ and a sink of strength $2m$ is placed at the origin. Show that the stream lines are the curves $(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$, where λ is a variable parameter. **[20 Marks]**

1998

97. Two particles in a plane are connected by a rod of constant length and are constrained to move in such a manner that the velocity of the middle of the rod is in the direction of the rod write down the equations of the constants. Is the system holonomic or non holonomic? Give reason for your answer. **[20 Marks]**
98. Using Lagrange equations obtain the differential equations of motion of a free particle in spherical polar coordinates. **[20 Marks]**
99. A rod of length $2a$ is suspended by a string of length l attached to one end; if the string and rod revolve about the vertical with uniform angular velocity ω , and their inclinations to the vertical be α and β respectively, show that $\omega^2 = \frac{3g \tan \beta}{3l \sin \alpha + 4a \sin \beta}$. **[20 Marks]**
100. A particle of mass m is fixed to a point P of the rim of a uniform circular disc of centre O mass m and radius a . The disc is held, its plane vertical its lowest point in contact with a perfectly rough horizontal table and with OP inclined at 60° to the upward vertical and is then released. If the subsequent motion continues in the same vertical plane show that, when OP makes an angle θ with the upward vertical.

$a(7 + 4\cos\theta)\theta^2 = 2g(1 - 2\cos\theta)$. Show that when OP is first horizontal, the acceleration of σ is $\frac{18}{49}g$.

[30 Marks]

101. Three equal uniform rods AB, BC, CD each of mass m and length $2a$ are at rest in a straight line smoothly jointed at B and C. A blow J is given to the middle rod at a distance x from its centre σ in a direction perpendicular to it; show that the initial velocity of σ is $\frac{2J}{3m}$ and that the initial angular velocities of the are

$$\frac{5a + 9x}{10ma^2} J, \frac{6x}{5ma^2} J, \frac{5a - 9x}{10ma^2} J$$

[30 Marks]

102. Show that a fluid of constant density can have a velocity \vec{q} given by $\vec{q} = \left[\frac{-2xyz}{(x^2 + y^2)^2}, \frac{(x^2 - y^2)z}{(x^2 + y^2)^2}, \frac{y}{x^2 + y^2} \right]$.

[20 Marks]

103. Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d ; If v and the corresponding velocities of the steam and if the motion be supposed to be that of divergence from the vertex of the cone, prove that $\frac{v}{V} = \left(\frac{D}{d}\right)^2 e^{\frac{v^2 - V^2}{2k}}$ where k the pressure is divided by the density, and supposed constant.

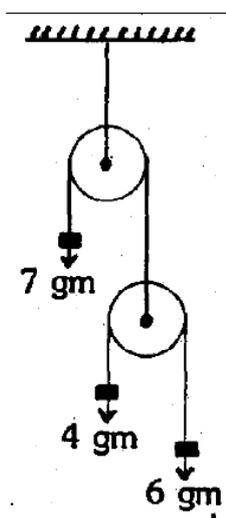
[20 Marks]

104. Between two fixed boundaries $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$ there is a two-dimensional liquid motion due to a source of strength m at the point $(r = a, \theta = 0)$ and an equal sink at the point $(r = b, \theta = 0)$. Show that the stream functions is $-m \tan^{-1} \left(\frac{r^4(a^4 - b^4)\sin 4\theta}{r^8 - r^4(a^4 + b^4)\cos 4\theta + a^4b^4} \right)$.

[20 Marks]

1997

105. Using Lagrange's equation obtain the differential equations for planetary motion. [20 Marks]
106. A circular, cylinder of radius 3 cm and weight W whose centre of gravity, G , is at a distance 1 cm from the axis, rolls on a horizontal plane. If motion be just started from the position of unstable equilibrium, show that the normal reaction of the plane when G , is in its lowest position is $W(2 + k^2)(1 + k^2)$, where $2k$ is the radius of gyration about an axis passing through G . [20 Marks]
107. A pulley system is given as shown, in the diagram. Discuss the motion of the system using Lagrange's method when the pulley wheels have negligible masses and moments of inertia and their wheels are friction less.



[20 Marks]

108. A solid homogeneous sphere is rolling on the inside of a hollow sphere the centers being always in the same vertical plane. Show that the smaller sphere will make complete revolution if, when it is in lowest position, the pressure on it is greater than $\frac{34}{7}$ times its own weight. **[30 Marks]**
109. Three equal uniform rods AB, BC, CD are smooth joined B and C and the ends A and D are fastened to smooth fixed point whose distance apart is equal to the length of the rod, the frame being at rest in the form of the square. A blow J is given perpendicular to AB at its middle point and in the plane square show that $\frac{3J^2}{40m}$ where m is the energy set up is where m is the mass of each rod. Find also the blows at the joined B and C . **[30 Marks]**
110. Show that $\frac{x^2}{a^2} \tan^2 t + \frac{y^2}{b^2} \cot^2 t = 1$ is a possible form of the bounding surface of a liquid and find expression for the normal velocity. **[20 Marks]**
111. A stream in a horizontal pipe after passing a contraction in the pipe, at which its cross-sectional area is A , is delivered at the atmospheric pressure at a place where the cross-sectional area is B . Show that if a side tube is connected with the pipe at the former place water will be sucked up through it into pipe from a reservoir at a depth $\frac{s^2}{2g} \left(\frac{1}{A^2} - \frac{1}{B^2} \right)$ below the pipe where s is the delivery per second. **[20 Marks]**
112. Using the method of images prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$ the solution is usual notations $\phi + i\psi = -m \log[(z^3 - z_0^3)(z^3 - z_0'^3)]$ where $z_0 = x_0 + iy_0$ and $z_0' = x_0' - iy_0'$. **[20 Marks]**
- 1996**
113. A uniform rod OA of length $2a$ free to turn about end O revolved with uniform angular velocity ω about the vertical OZ through O and is inclined at a constant angle α to OZ show that the value of α is either zero or $\cos^{-1} \left(\frac{3g}{4a\omega^2} \right)$. **[20 Marks]**
114. Six equal uniform rods form a regular hexagon loosely jointed at the angular points rests on a smooth table a blow is given perpendicular to one of them at its middle point. Show that the opposite rod begins to move with one tenth of the velocity of the rod that is struck. **[20 Marks]**
115. A cylinder of mass m radius R and moment of inertia I about its geometrical axis rolls down a hill without slipping under the action of gravity. If the velocity of the center of mass of the cylinder is initially v_0 . Find the velocity after the cylinder has dropped through a vertical distance h . **[20 Marks]**
116. A perfectly rough circular hoop of diameter 24 cm roll on a horizontal floor with velocity $V \frac{cm}{sec}$ toward an inelastic step of height 4 cm, the plane of the hoop being vertical and perpendicular to the edge of the step. Prove that hoop can mount the step without losing contact at any stage if $2.4\sqrt{2g} > V > 2.4\sqrt{g}$. **[30 Marks]**
117. A homogeneous sphere rolls down an imperfectly rough fixed sphere, starting from rest at the highest point. If the sphere separates when the line joining their centers makes an angle with the vertical, show that the coefficient of friction μ satisfies the following equation $e^{\mu\pi/3} = \frac{3\sqrt{3} + 6\mu}{4(1 - 2\mu^2)}$. **[30 Marks]**

118. Show that the motion specified by $\vec{q} = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$ is a possible flow for an incompressible fluid. Determine the stream lines. Show that the motion is irrotational and find the velocity potential. **[20 Marks]**
119. A sphere is at rest in an infinite mass of homogeneous liquid of density ρ . The pressure at infinity being \bar{w} show that if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is **[20 Marks]**
120. Find the stream function of two-dimensional motion due to two equal sources and an equal sink situated midway between them. In a region bounded by a fixed quadrantal arc and its radii deduce the motion due to a source and an equal sink situated at the end of one of the bounding radii. Show that the stream line leaving either end at an angle $\frac{\pi}{6}$ with the radius is $r^2 \sin\left(\frac{\pi}{6} + \theta\right) = a^2 \sin\left(\frac{\pi}{6} - \theta\right)$ where a is the radius of the quadrant. **[20 Marks]**

1995

121. How do you characterize
 (i) the simplest dynamical system?
 (ii) The most general dynamical system?
 Show that the equations of motion $\frac{d}{dt} \frac{\partial L}{\partial q_k} - \frac{\partial T}{\partial q_k} = Q_k, (k = 1, 2, \dots, n)$ correspond to a non-conservative but scleronomous system with n degrees of freedom where q, q_k, Q_k are respectively the generalized coordinates, the generalized velocities and the generalized forces. **[20 Marks]**
122. A solid uniform sphere has a light rod rigidly attached to it which passes through the center. The rod is so joined to a fixed vertical axis that the angle θ between the rod and the axis may alter but the rod must turn with the axis. If the vertical axis be forced to revolve constantly with uniform angular velocity, show that θ^2 is of the form $n^2(\cos \theta - \cos \beta)(\cos \alpha - \cos \theta)$ where n, α, β are certain constants. **[20 Marks]**
123. A uniform rod of length 20 cm which has one end attached to a fixed point by a light inextensible string of length $4\frac{1}{6}$ is performing small oscillation in a vertical plane about its position of equilibrium. Find its position at any time and the periods of principal oscillations. **[20 Marks]**
124. A carriage is placed on an inclined plane making an angle α with the horizon and rolls down without slipping between the wheels and the plane. The floor of the carriage is parallel to the plane and a perfectly rough ball is placed freely on it. Show that the acceleration of the carriage down the plane and a perfectly rough ball is placed freely on it show that the acceleration of the carriage down the plane is $\frac{14M + 4m' + 14m}{14M + 4m' + 21m} g \sin \alpha$ where M is the mass of the carriage excluding the wheels, m the sum of masses of the wheels which are uniform circular disc and M that of the ball which is a homogeneous solid sphere (the friction between the wheels and the axes is neglected) show that for the motion to be possible the coefficient of friction between the wheels and the plane must exceed the constant $\frac{7(M + m) + 2M'}{14M + 21m + 4M'} \tan \alpha$. **[30 Marks]**
125. A sphere of radius a is projected up an inclined with velocity V and angular velocity Ω in the sense which would cause it to roll up; if $V > a\Omega$ and the coefficient of friction greater than $\frac{2}{7} \tan \alpha$ show that the sphere will cease to ascend at the end of a time $\frac{5V + 2a\Omega}{5g \sin \alpha}$ where α is the inclination of the plane. **[30 Marks]**
126. Determine the restrictions on f_1, f_2, f_3 if $f_1(t) \frac{x^2}{a^2} + f_2(t) \frac{y^2}{b^2} + f_3(t) \frac{z^2}{c^2} = 1$ is a possible boundary surface of a liquid. **[20 Marks]**

127. If a, b, c, d, e, f are arbitrary constants, what type of fluid motion does the velocity $(a + by - cz, d - bx + ez, f + cx - ey)$ represent? [20 Marks]
128. If the fluid fill the region of spaces on the positive side of x -axis, which is a rigid boundary and if there be a source $+m$ at the point $(0, a)$ and an equal sink at $(0, b)$ and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, Show that the resultant pressure on the boundary is $\pi\rho m^2 \frac{(a-b)^2}{ab(a+b)}$, where ρ is the density of the fluid. [20 Marks]

1994

129. What is D' Alembert's principle? An inextensible string of negligible mass hanging over a smooth page at A connects the mass m_1 on a frictionless inclined of angle θ to another mass m_2 . Use D' Alembert's principle to prove that the mass will be in equilibrium if $m_2 = m_1 \sin \theta$. [20 Marks]
130. Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in smooth table so that rests on the table so that m_1 rests on the table surface and hangs suspended. Assuming m_2 moves only in a vertical line what are the generalized coordinates of the system? Write down Lagrange's equations of motion and obtain a first integral of the equations of motion. [20 Marks]
131. Twelve equal uniform rods are smoothly joined at their ends so as to form cubical framework, which is suspended from a point by string tied to one corner and kept in shape by a light string occupying the position of a vertical diagonal. Suppose that the string supporting the framework is cut, so that it falls and strikes a smooth inelastic horizontal plane. Find the impulsive reaction of the plane. [20 Marks]
132. A small light ring is threaded on a fixed thin horizontal wire. One end of a uniform rod of mass m and length $2a$ is freely attached to the ring. The coefficient of friction between the ring and the wire is μ the system is released from rest when the rod is horizontal and the vertical plane containing the wire. If the ring slips on the wire when the rod has turned through an angle α then prove that $\mu(10 \tan^2 \alpha + 1) = 9 \tan \alpha$. [30 Marks]
133. A uniform rod AB held at an inclination α to the vertical with one end A in contact with a rough horizontal table. If released, then prove that the rod will commence to slide at once if the coefficient of the friction is less than $\frac{3 \sin \alpha \cos \alpha}{1 + 3 \cos^2 \alpha}$. [30 Marks]
134. The particle velocity for a fluid motion referred rectangular axes is given by $\left(A \cos \frac{\pi x}{2a} \cos \frac{\pi z}{2a}, 0, A \sin \frac{\pi x}{2a} \sin \frac{\pi z}{2a} \right)$ where A, a are constants. Show that this is possible motion of an incompressible fluid under no body forces in an infinite fixed right tube $-a \leq x \leq a, 0 \leq z \leq 2a$ Also find the pressure associated with this velocity field. [20 Marks]
135. Determine the stream lines and the path lines of the particles when velocity field is given by $\left(\frac{x}{1+t}, \frac{y}{1+t}, \frac{z}{1+t} \right)$ [20 Marks]
136. Between the fixed boundaries $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$, there is a two-dimensional liquid motion due to a source of strength m at the point $r = a, \theta = 0$ and an equal sink at the point $r = b, \theta = 0$. Use the method of images to show that the stream function is $-m \tan^{-1} \left\{ \frac{r^4 (a^4 - b^4) \sin 4\theta}{r^8 - r^4 (a^4 + b^4) \cos 4\theta + a^4 b^4} \right\}$, Show also that the velocity at (r, θ) is $\frac{4m(a^4 - b^4)r^3}{(r^8 - 2a^4 r^4 \cos 4\theta + a^8)^{1/2} (r^8 - 2b^4 r^4 \cos 4\theta + b^8)^{1/2}}$ [20 Marks]

1993

137. Consider the two dynamical systems:
 (i) A sphere rolling down from the top of a fixed sphere
 (ii) A cylinder rolling without slipping down a rough inclined plane.
 (iii) State whether (I) is rheonomic, holonomic and justify your claim
 (iv) Give reasons for (II) to be classified as scleronomic holonomic Is it conservative? **[20 Marks]**
138. Find
 (i) the Lagrangian
 (ii) The equation of motion for the following system:
 A particle is constrained to move in a plain under the influence of an attraction towards to origin proportional to the distance from it and also of a force perpendicular to the radius vector inversely proportional to the distance of the particle from the origin in anticlockwise direction. **[20 Marks]**
139. A heavy uniform rod rotating in a vertical plane fall and strikes a smooth inelastic horizontal plane. Find the impulse. **[20 Marks]**
140. The door of a railway carriage has its hinges, supposed smooth, towards the engine which start with an acceleration f . Prove that the door closed in time $\left(\frac{a^2 + K^2}{2af}\right)^{1/2} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}}$ with an angular velocity $\sqrt{\frac{2af}{a^2 + K^2}}$ where $2a$ is the breadth of the door and K its radius of gyration about a vertical axis through G , the center of mass. **[30 Marks]**
141. A solid homogeneous sphere is resting on the top another fixed sphere and roll down it. Write down the equation of smooth and find the friction when does the upper sphere leave the lower sphere if
 (i) both the spheres are smooth
 (ii) The upper sphere is sufficiently rough so not to slip. **[30 Marks]**
142. Show that $u = \frac{-2xyz}{(x^2 + y^2)^2}, v = \frac{(x^2 - y^2)z}{(x^2 + y^2)^2}, w = \frac{y}{x^2 + y^2}$ are the velocity components of a possible liquid motion. Is this motion irrotational? **[20 Marks]**
143. Steam is rushing from a boiler through conical pipe the diameter of the ends of which are D and d ; if V and v be the corresponding velocities of the steam and if the motion be supposed to be that of divergence from the vertex of the cone, prove that $\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2 - V^2)/2K}$ where K is the pressure divided by the density and supposed constant. **[20 Marks]**
144. Prove that for liquid circulating irrotationally in part of the plane between two non-intersecting circle the curve of constant velocity are Cassini's ovals. **[20 Marks]**

1992

145. Classify each of the following dynamical systems according as they are
 a. scleronomic or rheonomic
 b. holonomic or non-holonomic
 c. conservative or non-conservative
 [I] a horizontal cylinder of radius a rolling inside a perfectly rough hollow horizontal cylinder of radius $b > a$

- [II] A particle constrained to move along a line under the influence of a force which is inversely proportional to the square of its distance from a fixed point and a damping force proportional to the square of its distance from a fixed point and a damping force proportional to the square of the instantaneous speed.
- [III] a particle moving on a very long frictionless wire which rotates with constant angular speed about a horizontal axis.

[10 Marks]

146. When the Lagrangian function has the form $L = q_k \dot{q}_k - \sqrt{1 - q_k^2}$. Show that generalized acceleration is zero. [20 Marks]

147. The end of a uniform rod AB of length $2a \cos 15^\circ$ and weight W are constrained to slide on a smooth circular wire of radius a fixed with its plane vertical. The end A is connected by an elastic string of natural length a and modulus of elasticity $W/2$ to the highest point of the wire. If θ is the angle which the perpendicular bisector of the rod makes with the downward vertical, show that the potential energy V is given by

$$V = -\frac{W_a}{2} \left\{ \cos(\theta - 75^\circ) + 2 \cos \frac{1}{2}(\theta + 75^\circ) \right\} + \text{constant}$$

verify that $\theta = 25^\circ$ defines a position of equilibrium and investigate its stability. [30 Marks]

148. A uniform rod of length $2a$ which has one end attached to a fixed point by a light inextensible string of length $\frac{5a}{12}$ performing small oscillations in a vertical plane about its position of equilibrium. Find the position at any

time show that the periods of its principal oscillations are $2\pi \sqrt{\frac{5a}{3g}}$ and $\pi \sqrt{\frac{a}{3g}}$. [30 Marks]

149. A uniform circular disc of radius a and mass m rolls down a rough inclined plane without sliding. Show that the center of the disc moves with contact acceleration $2/3 g \sin \alpha$ and the coefficient of friction $\mu > 1/3 \tan \alpha$ where α is the inclination of the plane. [30 Marks]

150. Show that the variable ellipsoid $\frac{x^2}{a^2 k^2 t^4} + kt^2 \left[\left(\frac{y}{b} \right)^2 + \left(\frac{z}{c} \right)^2 \right] = 1$ is a possible form the bounded surface of a liquid motion at any time t . [20 Marks]

151. Find the lines of flow in the 2-dimensional fluid motion given by $\phi + i\psi = -\frac{1}{2} n(x + iy)^2 e^{2int}$. Prove or verify that the paths of the particles of the fluid (in polar coordinates) may be obtained by eliminating t from the equations $r \cos(nt + \theta) - x_0 = r \sin(nt + \theta) - y_0 = nt(x_0 - y_0)$. [20 Marks]

152. A source of strength m and a vortex of strength k are placed at the origin of the 2-dimensional motion of unbounded liquid prove that the pressure at infinity exceeds that pressure at distance r from the origin by $\frac{1}{2} \frac{(m^2 + k^2)}{r^2} \rho$. [20 Marks]