RAMANASRI

Learning on time

IAS/IFoS Maths Optional

2023 UPSC
Maths Optional
Paper - 1 Solutions

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UPSC 2023
ESSAY TOPIC
125 MARKS

MATHEMATICS IS THE MUSIC OF REASON

-JAMES JOSEPH SYLVESTER

Q. 1 a) Let $V_1 = (2, -1, 3, 2)$, $V_2 = (-1, 1, 1, -3)$ and $V_3 = (1, 1, 9, -5)$ be three vectors of the space $\mathbb{R}^4$. Does $(3, -1, 0, -1) \in$ span $\{V_1, V_2, V_3\}$? Justify your answer.

Sol^n: Let

$$(3, -1, 0, -1) = x(2, -1, 3, 2) + y(-1, 1, 1, -3)$$
$$+ z(1, 1, 9, -5)$$
$$= (2x - y + z, \ -x + y + z, \ 3x + y + 9z, \ 2x - 3y - 5z)$$

where $x, y, z \in \mathbb{R}$ — ①

$$\left.\begin{aligned} 2x - y + z &= 3 \\ -x + y + z &= -1 \\ 3x + y + 9z &= 0 \\ 2x - 3y - 5z &= -1 \end{aligned}\right\} \ ②$$

Now $AX = B$

where $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$$[A/B] = \begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}$$

$$R_1 \longleftrightarrow R_2$$

$$\Rightarrow [A|B] = \begin{bmatrix} -1 & 1 & 1 & | & -1 \\ 2 & -1 & 1 & | & 3 \\ 3 & 1 & 9 & | & 0 \\ 2 & -3 & -5 & | & -1 \end{bmatrix}$$

$$R_1 \rightarrow (-1) R_1$$

$$\Rightarrow [A|B] = \begin{bmatrix} 1 & -1 & -1 & | & 1 \\ 2 & -1 & 1 & | & 3 \\ 3 & 1 & 9 & | & 0 \\ 2 & -3 & -5 & | & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1, \quad R_4 \rightarrow R_4 - 2R_1$$

$$\Rightarrow [A|B] = \begin{bmatrix} 1 & -1 & -1 & | & 1 \\ 0 & 1 & 3 & | & -5 \\ 0 & 4 & 12 & | & -3 \\ 0 & -1 & -3 & | & -3 \end{bmatrix}$$

$$R_3 \rightarrow \left(\frac{1}{4}\right) R_3 \quad R_4 \rightarrow (R_4 + R_2)$$

$$\Rightarrow [A|B] = \begin{bmatrix} 1 & -1 & -1 & | & 1 \\ 0 & 1 & 3 & | & -5 \\ 0 & 1 & 3 & | & -3/4 \\ 0 & 0 & 0 & | & -8 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\Rightarrow [A|B] = \begin{bmatrix} 1 & -1 & -1 & | & 1 \\ 0 & 1 & 3 & | & -5 \\ 0 & 0 & 0 & | & 17/4 \\ 0 & 0 & 0 & | & -8 \end{bmatrix}$$

Clearly, $\rho(A) = 2$, $\rho(A|B) = 4$ $\therefore$ $\boxed{\rho(A) < \rho(A|B)}$

$\therefore$ The given equation has no solution.

Hence, $(3, -1, 0, -1) \notin \text{span} \{v_1, v_2, v_3\}$

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Topic

Remarks

Q. 1. b) Find the rank and nullity of the linear transformation:

$T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$T(x, y, z) = (x+z, x+y+2z, 2x+y+3z)$

Soln: Given $T: \mathbb{R}^3 \to \mathbb{R}^3$ and given linear transformation

$T(x, y, z) = (x+z, x+y+2z, 2x+y+3z)$

i) For null space

$\qquad T(x, y, z) = 0$

$(x+z, x+y+2z, 2x+y+3z) = 0$

$\qquad x + z = 0 \qquad - ①$

$\qquad x + y + 2z = 0 \qquad - ②$

$\qquad 2x + y + 3z = 0 \qquad - ③$

from ①, $x = -z$,

② $\Rightarrow$ $z + y - 2x = 0 \Rightarrow x = y$

Hence we get the null space

$N(T) = \{ x(1, 1, -1) \mid x \in \mathbb{R} \}$

$\therefore$ $\boxed{\text{Nullity} = 1}$ (as no. of vector in basis of $N(T)$)

ii) By Rank - Nullity theorem,

$\qquad \dim \mathbb{R}^3 = \text{rank}(T) + \text{Nullity}(T)$

$\qquad 3 = \text{rank}(T) + 1$

$\qquad \therefore \boxed{\text{Rank}(T) = 2}$

Hence $\boxed{\text{Rank}(T) = 2 \ \& \ \text{Nullity}(T) = 1}$

Q. 1 c) Find the values of p and q for which

$$\lim_{x \to 0} \frac{x(1+p\cos x) - q\sin x}{x^3} \quad \text{exists and}$$

equals 1.

Sol^n:

$$\lim_{x \to 0} \frac{x(1+p\cos x) - q\sin x}{x^3} \quad \left(\frac{0}{0}\right) \quad \begin{array}{l} \text{Indeterminate} \\ \text{Fundamental} \\ \text{form} \end{array}$$

∴ Applying L'Hospital's rule

$$\lim_{x \to 0} \frac{\frac{d}{dx}\left[x(1+p\cos x) - q\sin x\right]}{\frac{d}{dx}(x^3)} = 1$$

∴ $$\lim_{x \to 0} \frac{1 + p\cos x - px\sin x - q\cos x}{3x^2} = 1$$

∴ $$\lim_{x \to 0} \frac{1 + (p-q)\cos x - xp\sin x}{3x^2} = 1$$

to satisfy limit $1 + p - q = 0$

$$\Rightarrow \boxed{p+1 = q} \quad -①$$

Applying L'Hospital's rule

$$\lim_{x \to 0} \frac{-(p-q)\sin x - p\sin x - xp\cos x}{6x} = 1$$

$$\lim_{x \to 0} \frac{(q-p)\sin x - p\sin x - xp\cos x}{6x} = 1$$

$$\lim_{x \to 0} \frac{(q-2p)\sin x - px\cos x}{6x} = 1$$

$$\lim_{x \to 0} \frac{(q-2p)}{6} \cdot \frac{\sin x}{x} - \lim_{x \to 0} \frac{p}{6}\cos x = 1$$

$$\Rightarrow \quad \frac{q-2p}{6} - \frac{p}{6} = 1$$

$$\Rightarrow \quad q - 3p = 6$$

$$q = 3p + 6$$

$$\boxed{q = 3(p+2)} \quad - \text{②}$$

$$p - q = -1$$
$$- \quad \underline{3p - q = -6}$$
$$-2p = 5$$

$$\therefore \boxed{p = -\frac{5}{2}} \quad \text{and} \quad \begin{array}{c} q = 1 - 5/2 \\ \boxed{q = -\frac{3}{2}} \end{array}$$

Q. 1 d) Examine the convergence of the integral

$$\int_0^1 \frac{\log x}{1+x} \, dx$$

Sol^n:

Method 1:

$$f(x) = \frac{\log x}{1+x} \, dx$$

Clearly, f is unbounded at $x=0$

$$I = \int_0^1 \frac{\log x}{1+x} \qquad - \text{①}$$

Only point of infinite discontinuity is

$$\underline{x=0}$$

$$\therefore \lim_{x \to 0} x^\mu f(x) = \lim_{x \to 0} x^\mu \frac{\log x}{1+x} = 0, \quad \text{if } \mu > 0$$

Hence we choose $0 < \mu < 1$, then by

μ test $\underline{I}$ is convergent.

Method 2:

Since $\frac{\log x}{1+x}$ is negative on $(0,1]$,

we take $f(x) = \frac{-\log x}{1+x}$

Here '0' is the only point of infinite

discontinuity of f on $[0,1]$

take $g(x) = \frac{1}{x^n}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \frac{-x^n \log x}{1+x} = 0, \text{ if } n > 0$$

Taking n between 0 and 1, the integral

$$\int_0^1 g(x) \, dx \text{ is convergent.}$$

$\therefore$ By comparison test.

$$\int_0^1 f(x) \, dx \text{ is convergent.}$$

Hence $\boxed{\int_0^1 \dfrac{\log x}{1+x} \, dx \text{ is convergent}}$

Q. 1 e) A variable plane which is at a constant distance $3p$ from the origin O cuts the axes in the points A, B, C respectively. Show that the locus of the centroid of the tetrahedron $OABC$ is

$$9\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = \frac{16}{p^2}$$

Sol^n:

Let the equation of the variable plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad - ①$$

It is given that this plane is at a distance '$3p$' from $(0,0,0)$

$$\therefore \quad 3p = \frac{1}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}}$$

or

$$\frac{1}{9p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \qquad - ②$$

Also the plane ① meets the axes in A, B and C. So the co-ordinates O, A, B and C are $(0,0,0), (a,0,0), (0,b,0)$ and $(0,0,c)$ respectively.

Let (x, y, z) be the centroid of the tetrahedron $OABC$, then

$$x = \frac{1}{4}(0 + a + 0 + 0)$$

$$x = \frac{1}{4}a.$$

Similarly $y = \frac{1}{4}b$ and $z = \frac{1}{4}c$

or $a = 4x$, $b = 4y$, $c = 4z$

Substituting these values of a, b and c in ②, we have the required locus as

$$\frac{1}{9p^2} = \frac{1}{16x^2} + \frac{1}{16y^2} + \frac{1}{16z^2}$$

or

$$\boxed{9\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = \frac{16}{p^2}}$$

Hence proved.

Q. 2 a) If the matrix of a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ is $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

then find the matrix of T relative to the basis $\{(1,1,1), (0,1,1), (0,0,1)\}$

Sol^n: Let $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

given that $[T]_B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

let any $\alpha(x,y,z) \in \mathbb{R}^3$ can be expressed as

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$T(x,y,z) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$

$\qquad\qquad\qquad\qquad\qquad\qquad - ①$

$\left.\begin{array}{l} T(1,0,0) = (1,-1,0) \\ T(0,1,0) = (1,2,1) \\ T(0,0,1) = (2,1,3) \end{array}\right\}$ from $[T]_B$ given

$\therefore ① \Rightarrow T(x,y,z) = x(1,-1,0) + y(1,2,1) + z(2,1,3)$

$\boxed{T(x,y,z) = (x+y+2z, \; -x+2y+z, \; y+3z)}$

is required transformation

Let $B' = \{(1,1,1), (0,1,1), (0,0,1)\}$

Let $(x,y,z) = a(1,1,1) + b(0,1,1) + c(0,0,1)$

$\qquad (x,y,z) = (a, \; a+b, \; a+b+c)$

$$\therefore \quad a = x$$

$$a + b = y \quad \Rightarrow \quad b = y - x$$

$$a + b + c = z \quad \Rightarrow \quad c = z - y$$

$$\therefore (x, y, z) = x(1, 1, 1) + (y-x)(0, 1, 1)$$
$$+ (z-y)(0, 0, 1)$$

We need to express with respect to B'

$$T(1, 1, 1) = (4, 2, 4) = 4(1,1,1) - 2(0,1,1) + 2(0,0,1)$$

$$T(0, 1, 1) = (3, 3, 4) = 3(1,1,1) + 0(0,1,1) + 1(0,0,1)$$

$$T(0, 0, 1) = (2, 1, 3) = 2(1,1,1) - 1(0,1,1) + 2(0,0,1)$$

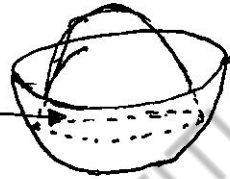Hence matrix of linear transformation is

$$[T]_{B'} = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

is required solution.

Q. 2. b) Evaluate the triple integral which gives the volume of the solid enclosed between the two paraboloids $Z = 5(x^2 + y^2)$ and $Z = 6 - 7x^2 - y^2$.

Sol^n:

To find volume of solid enclosed bet^n these two paraboloids

The bounded region of the above paraboloids is

$$Z = 5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

$$12x^2 + 6y^2 = 6$$

$$2x^2 + y^2 = 1$$

is intersection of two curves.

Thus volume of region

$$V = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} 1 \, dx \, dy \, dz \quad —①$$

Proceeding with cylindrical co-ordinate

$$x = \frac{r}{\sqrt{2}} \cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$dx \, dy \, dz = \frac{r}{\sqrt{2}} \, dr \, d\theta \, dz$$

On reducing the equation ①

$$V = \int\limits_{0}^{2\pi} \int\limits_{0}^{1} \int\limits_{\frac{5}{2}r^2\cos^2\theta + 5r^2\sin^2\theta}^{6-\frac{7}{2}r^2\cos^2\theta - r^2\sin^2\theta} 1 \cdot \frac{r}{\sqrt{2}} \, dr \, d\theta \, dz$$

$$= \frac{1}{\sqrt{2}} \int\limits_{0}^{2\pi} \int\limits_{0}^{1} (6 - 6r^2) \, r \, dr \, d\theta$$

$$= \frac{1}{\sqrt{2}} \, 2\pi \int\limits_{0}^{1} (6 - 6r^2) \, r \, dr$$

$$= \frac{1}{\sqrt{2}} \, 2\pi \left(\frac{6r^2}{2} - \frac{6r^4}{4} \right)_{0}^{1}$$

$$= \frac{2\pi}{\sqrt{2}} \left(3 - \frac{3}{2} \right)$$

$$= \frac{2\pi}{\sqrt{2}} \cdot \frac{3}{2}$$

$$= \frac{3\pi}{\sqrt{2}}$$

$$\therefore \boxed{V = \frac{3\pi}{\sqrt{2}}}$$

Q. 2 c) i) Show that the equation ~~2x²~~

$2x^2 + 3y^2 - 8x + 6y - 12z + 11 = 0$ represents an elliptic paraboloid. Also find its principal axis and principal planes.

Sol^n: $2x^2 + 3y^2 - 8x + 6y - 12z + 11 = 0$

$a = 2 \quad b = 3 \quad c = 0$

$f = 0 \quad g = 0 \quad h = 0$

$u = -4 \quad v = 3 \quad w = -6 \quad d = 11$

We know that the discriminating cubic equation is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$\lambda(\lambda-3)(\lambda-2) = 0$

$\lambda = 0, \quad \lambda = 2, \quad \lambda = 3$

Now putting $\lambda = 0$ in the determinant given by ① and associating each row with l_3, m_3, n_3 we have

$$l_3 = 0, \quad m_3 = 0, \quad n_3 \cdot 0 = 0 \quad n_3 \neq 0$$

because $l^2 + m^2 + n^2 = 1$

$$\therefore n = 1$$

Now $\quad k = u l_3 + v m_3 + w n_3$

$$k = -4 \times 0 + 3 \times 0 + -6 \times 1$$

$$\boxed{k = -6}$$

Required reduced eq^n is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2 k z = 0$$

$$2x^2 + 3y^2 - 12z = 0$$

$$\boxed{\frac{x^2}{3} + \frac{y^2}{2} = 2z}$$

which represent an elliptical paraboloid as both λ_1 and λ_2 are positive

Also if $F(x,y,z) = 0$ be the given surface then the co-ordinate of its vortex are given by solving any two of these equations.

$$\frac{\partial F / \partial x}{l_3} = \frac{\partial F / \partial y}{m_3} = \frac{\partial F / \partial z}{n_3} = 2k$$

and $k(l_3 x + m_3 y + n_3 z) + u x + v y + w z$
$$+ d = 0$$

$$F \equiv 2x^2 + 3y^2 - 8x + 6y - 12z + 11 = 0$$

$$\frac{\partial F}{\partial x} = 4x - 8 \;;\; \frac{\partial F}{\partial y} = 6y + 6 \;;\; \frac{\partial F}{\partial z} = -12$$

$$k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$$

$$-12(0 \cdot x + 0 \cdot y + 1 \cdot z) - 4x + 3y - 6z + 11 = 0$$

$$-12z - 4x + 3y - 6z + 11 = 0$$

$$-4x + 3y - 18z + 11 = 0$$

$$\frac{\frac{\partial F}{\partial x}}{l_3} = \frac{\frac{\partial F}{\partial y}}{m_3} = \frac{\frac{\partial F}{\partial z}}{n_3} = 2k$$

$$\frac{4x - 8}{0} = \frac{6y + 6}{0} = \frac{-12}{1} = 2(-12)$$

$$\Rightarrow 4x - 8 = 0 \;\bigg|\; 6y + 6 = 0 \;\bigg|\; -8 - 3 - 18z + 11 = 0$$

$$\Rightarrow x = 2 \;\bigg|\; y = -1 \;\bigg|\; -18z = 0$$

$$\Rightarrow z = 0$$

$$\therefore \text{Vertex is } (2, -1, 0)$$

Principal axis equation is
$$\frac{x - 2}{0} = \frac{y + 1}{0} = \frac{z - 0}{1}$$

Principal plane equation is given by
$$\lambda(lx + my + nz) + (ul + vm + wn) = 0$$

for $\lambda = 2$:

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix} \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$m_1 = 0 ; \quad n_1 = 0 \Rightarrow l_1 = 1$

$\lambda(l_1 x + m_1 y + n_1 z) + (u l_1 + v m_1 + w n_1) = 0$

$\therefore \quad 2(x) + (-4 \times 1) = 0$

$\Rightarrow \quad \boxed{x - 2 = 0}$

for $\lambda = 3$:

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{vmatrix} \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -l_2 = 0 ; \quad n_2 = 0 \Rightarrow m_2 = 1$

$\lambda(l_2 x + m_2 y + n_2 z) + (u l_2 + v m_2 + w n_2) = 0$

$\therefore \quad 3y + 3 = 0$

$\Rightarrow \quad \boxed{y + 1 = 0}$

$\therefore$ Required principal plane is

$$\boxed{\begin{array}{l} x - 2 = 0 \quad \& \\ y + 1 = 0 \end{array}}$$

Q. 2 c) ii) The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the co-ordinate axes in A, B, C respectively. Prove that the equation of the cone generated by the lines drawn from the origin O to meet the circle ABC is

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{b}{a} + \frac{a}{b}\right) = 0$$

Sol^n:

The plane ABC is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ — ①

It meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ & $C(0, 0, c)$.

Since the sphere intercepts a length 'a' on x-axis so it passes through the point $(a, 0, 0)$. Similarly it passes through the points $(0, b, 0)$ & $(0, 0, c)$. Also it passes through the origin i.e. $(0, 0, 0)$

Let the equation of sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad - ②$$

If it passes through $(0, 0, 0)$ then from ② we have $d = 0$ — ③

If ② passes through $(a, 0, 0)$, then we get $a^2 + 2ua + d = 0$

or from ③, $a^2 + 2ua + 0 = 0$

or $u = -\frac{1}{2}a$, as $a \neq 0$

Similarly as ② passes through $(0, b, 0)$ and $(0, 0, c)$ we get

$$v = -\frac{1}{2}b \quad \text{and} \quad w = -\frac{1}{2}c$$

Hence from ②, required equation is

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \quad - ④$$

The required cone is generated by the lines drawn from O to meet the circle ABC (given by ① and ④ together) and will be homogeneous. So making ④ homogeneous with the help of ①, we get the required equation as

$$x^2 + y^2 + z^2 - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

or $\boxed{yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{b}{a} + \frac{a}{b}\right) = 0}$

Hence proved.

Q. 3 a)   Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

i) Verify the Cayley-Hamilton theorem for the matrix A.

ii) Show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$, where I is the identity matrix of order 3. Hence find A^{40}.

Soln: i) Characteristic (Ch.) equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 1) = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Ch. eqn of A is $\lambda^3 - \lambda^2 - \lambda + 1 = 0$

By Cayley-Hamilton theorem

$$A^3 - A^2 - A + I = 0 \qquad - ①$$

L.H.S. $= A^3 - A^2 - A + I$

$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore$ L.H.S. $= A^3 - A^2 - A + I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= $ R. H. S.

Hence Cayley - Hamilton theorem is verified.

ii) By Principle of mathematical induction

① $n = 3 \quad A^3 = A + A^2 - I \quad$ is true by eq ①

② $n = k \quad A^k = A^{k-2} + A^2 - I \quad$ is assume to be true

③ $n = k + 1 \quad A^{k+1} = A^{k-1} + A^2 - I$

$L.H.S. = A^{k+1} = A^k \cdot A$

$\qquad = (A^{k-2} + A^2 - I) A$

$\qquad = (A^{k-1} + A^3 - A)$

$\qquad = A^{k-1} + A^2 + A - I - A$

$\qquad = A^{k-1} + A^2 - I$

$\qquad = R.H.S.$

Hence we proved

$$\boxed{A^n = A^{n-2} + A^2 - I} \quad \text{for } n \geqslant 3$$

To find A^{40} :-

$A^{40} = A^{38} + A^2 - I$

$\qquad = A^{36} + 2A^2 - 2I$

$\qquad = A^{34} + 3A^2 - 3I$

$\qquad \vdots$

$\qquad = A^2 + 19A^2 - 19I$

$A^{40} = 20A^2 - 19I$

$\qquad = \begin{bmatrix} 20 & 0 & 0 \\ 20 & 20 & 0 \\ 20 & 0 & 20 \end{bmatrix} - \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix}$

$\therefore \boxed{A^{40} = \begin{bmatrix} 1 & 0 & 0 \\ 20 & 1 & 0 \\ 20 & 0 & 1 \end{bmatrix}}$

$n = 4$

$A^4 = 2A^2 - I$

$A^8 = 4A^4 - 4A^2 + I$

$\qquad = 4(2A^2 - I) - 4A^2 + I$

$\qquad = 4A^2 - 3I$

$A^{16} = 16A^4 - 24A^2 + 9I$

$\qquad = 16(2A^2 - I) - 24A^2 + 9I$

$\qquad = 8A^2 - 7I$

$A^4 = 2A^2 - I$

$A^8 = 4A^2 - 3I$

$A^{16} = 8A^2 - 7I$

$A^{40} = 20A^2 - 19I$

is required solution

Q. 3. b) Justify whether $(0,0)$ is an extreme point for the function $f(x,y) = 2x^4 - 3x^2y + y^2$

Sol^n: Given $f(x,y) = 2x^4 - 3x^2y + y^2$

$p = \dfrac{\partial f}{\partial x} = 8x^3 - 6xy$ $\qquad r = \dfrac{\partial^2 f}{\partial x^2} = 24x^2 - 6y$

$q = \dfrac{\partial f}{\partial y} = -3x^2 + 2y$ $\qquad t = \dfrac{\partial^2 f}{\partial y^2} = 2$

$\qquad\qquad s = \dfrac{\partial^2 f}{\partial x \partial y} = -6x$

At point $(0,0)$:

$\left(\dfrac{\partial f}{\partial x}\right)_{(0,0)} = 8(0)^3 - 6(0)(0) = 0$

$\left(\dfrac{\partial f}{\partial y}\right)_{(0,0)} = -3(0)^2 + 2(0) = 0$

$\left(\dfrac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = 24(0)^2 - 6(0) = 0$

$\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = -6(0) = 0$

$\left(\dfrac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 2$

Now, $rt - s^2 = 0 \times 2 - (0)^2 = 0$

$\qquad\qquad$ and $r = 0$ at $(0,0)$

∴ (0,0) does not meet the condition for extreme point using second partial derivative.

It is a <u>critical point</u> <u>but not an</u> <u>extreme point</u> for the given function.

Hence <u>above function neither have</u> <u>maxima nor minima</u>.

Q. 3 c) Find the equation of the sphere through the circle $x^2 + y^2 + z^2 - 4x - 6y + 2z - 16 = 0$; $3x + y + 3z - 4 = 0$ in the following cases.

i) the point $(1, 0, -3)$ lies on the sphere.

ii) the given circle is a great circle of the sphere.

Sol^n: We know that, equation of the sphere passing through the circle $s = 0 = P$ is

$$s + \lambda P = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 4x - 6y + 2z - 16 + \lambda(3x + y + 3z - 4) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + (3\lambda - 4)x + (\lambda - 6)y + (3\lambda + 2)z - 4\lambda - 16 = 0 \quad - \text{①}$$

i) But given that above sphere passes through the point $(1, 0, -3)$

$$\Rightarrow (1 + 9 - 4 - 6 - 16) + \lambda(3 - 9 - 4) = 0$$

$$-16 + \lambda(-10) = 0$$

$$\lambda = \frac{-16}{10} = \frac{-8}{5}$$

∴ Sphere equation

$$x^2 + y^2 + z^2 - 4x - 6y + 2z - 16 - \frac{8}{5}(3x + y + 3z - 4) = 0$$

$$5x^2 + 5y^2 + 5z^2 - 44x - 38y - 40z - 48 = 0$$

$$\boxed{x^2 + y^2 + z^2 - \frac{44}{5}x - \frac{38}{5}y - 8z - \frac{48}{5} = 0}$$

Q. 4. a) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

by reducing it to row-reduced echelon form.

Sol^n:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$R_2 \longleftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix}$$

$R_2 \to (-1) R_2$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix}$$

$$R_3 \to R_3 + 3R_2 \,, \quad R_4 \to R_4 - 5R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

$$R_3 \to (-1)$$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

$$R_4 \to R_4 - 9R_3$$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Above matrix is in echelon form. Hence rank of matrix A is non-zero row in echelon form

$$\boxed{\rho(A) = 3}$$

Q. 4 b) Trace the curve $y^2(x^2-1) = 2x-1$

Sol^n: Steps :-

1. Curve is symmetrical about x-axis

2. It does not pass through the origin.

3. Curve meets x-axis in $\left(\frac{1}{2}, 0\right)$ and
   Y-axis in $(0, 1)$ and $(0, -1)$
   Line $x = \frac{1}{2}$ is tangent to curve at $\left(\frac{1}{2}, 0\right)$

4. $x = \pm 1$, $y = 0$ are its only asymptote

5. $y^2 = \dfrac{2x-1}{x^2-1}$

   $\Rightarrow$ for $x < -1$ and $\frac{1}{2} < x < 1$, y is imaginary

   $\therefore$ Curve lies between region $-1 < x < \frac{1}{2}$ & $x > 1$

6. $y = \pm \sqrt{\dfrac{2x-1}{x^2-1}}$

   $\Rightarrow \dfrac{dy}{dx} = \pm \left(\dfrac{-x^2+x+1}{(2x-1)^{1/2} (x^2-1)^{3/2}} \right)$

   for $x > 1$, $\dfrac{dy}{dx} < 0$

   $\Rightarrow$ y is decreasing for $x > 1$

   for $-1 < x < \frac{1}{2}$, $\dfrac{dy}{dx} < 0$
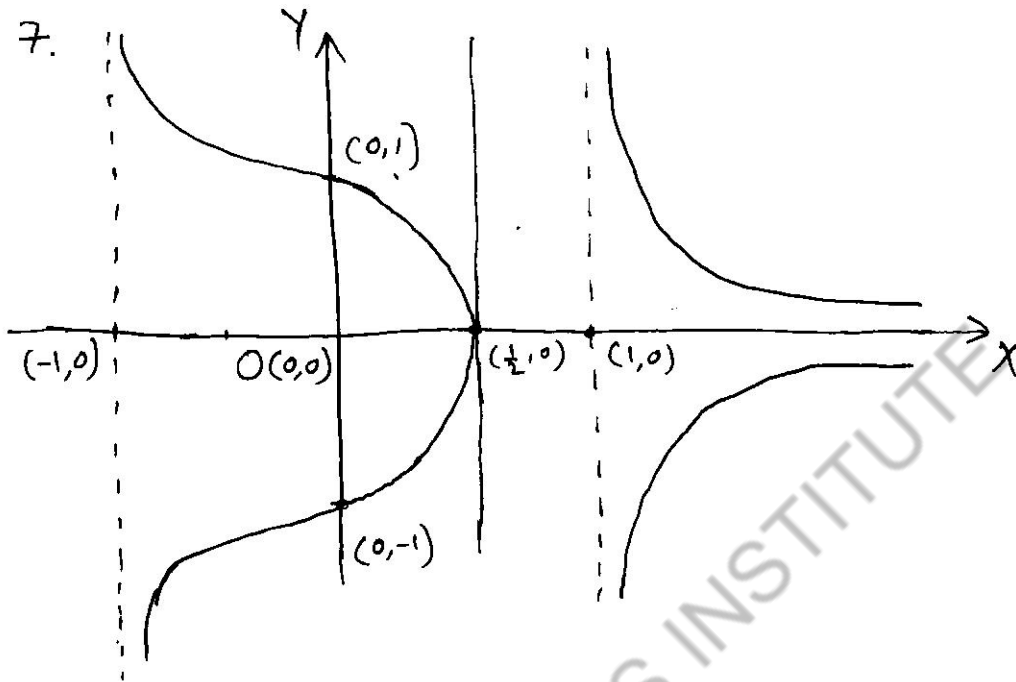
   $\Rightarrow$ y is decreasing in $\left[-1, \frac{1}{2}\right]$

7.

fig. Curve $y^2(x^2-1) = 2x-1$

Q. 4. c) Prove that the locus of a line which meets the lines $y = mx$, $z = c$; $y = -mx$, $z = -c$ and the circle $x^2 + y^2 = a^2$, $z = 0$ is

$$c^2 m^2 (cy - mzx)^2 + c^2 (yz - cmx)^2 = a^2 m^2 (z^2 - c^2)^2$$

Sol^n: The given lines are

$$y - mx = 0, \quad z - c = 0 \qquad - ①$$

$$y + mx = 0, \quad z + c = 0 \qquad - ②$$

and the circle is

$$x^2 + y^2 = a^2; \quad z = 0 \qquad - ③$$

Any line intersecting ① & ② is

$$\left. \begin{array}{l} y - mx + k_1 (z - c) = 0 \\ y + mx + k_2 (z + c) = 0 \end{array} \right\} - ④$$

If it meets the circle ③, we have to eliminate x, y, z from ③ & ④

putting $z = 0$ in ④, we get

$$y - mx + k_1 (-c) = 0$$

$$y + mx + k_2 (c) = 0$$

Solving $\dfrac{y}{-mk_2 c + mk_1 c} = \dfrac{x}{-ck_1 - ck_2} = \dfrac{1}{m + m}$

$$\Rightarrow \quad x = \dfrac{-(k_1 + k_2) c}{2m}$$

$$y = \dfrac{c(k_1 - k_2)}{2}$$

putting these values of x, y in ③, we get

$$\frac{c^2(k_1+k_2)^2}{4m^2} + \frac{c^2(k_1-k_2)^2}{4} = a^2$$

$$\Rightarrow \quad c^2(k_1+k_2)^2 + c^2m^2(k_1-k_2)^2 = 4a^2m^2 \quad -⑤$$

To find the locus,

eliminate k_1, k_2 from ④ & ⑤

$\therefore$ ④ $\equiv k_1 = \dfrac{-(y-mx)}{z-c} = \dfrac{mx-y}{z-c}$

$$k_2 = -\frac{(y+mx)}{z+c}$$

substituting these values in ⑤

$\therefore$ ⑤ $\Rightarrow c^2\left[\left(\dfrac{mx-y}{z-c}\right) + \left(\dfrac{-mx-y}{z+c}\right)\right]$

$$+ c^2m^2\left[\left(\frac{mx-y}{z-c}\right) + \left(\frac{mx+y}{z+c}\right)\right] = 4a^2m^2$$

on simplification we get

$$c^2m^2(cy-mzx)^2 + c^2(yz-cmx)^2 = a^2m^2(z^2-c^2)^2$$

which is the required locus.

Q 5 a) obtain the solution of the initial value problem $\frac{dy}{dx} - 2xy = 2$, $y(0) = 1$ in the form $y = e^{x^2}[1 + \sqrt{\pi}\; erf(x)]$.

soln —

$\frac{dy}{dx} - 2xy = 2$ is Linear differential eqn in y.

$P = -2x \qquad q = 2$

$IF = e^{\int p\, dx} = e^{\int -2x\, dx} = e^{-x^2}$

Hence solution is

$y \cdot IF = \int q\; IF\; dx + c$

$y \cdot e^{-x^2} = \int 2\, e^{-x^2}\, dx + c \qquad\qquad ①$

$y(0) = 1$

$\therefore \; 1 \cdot e^0 = \int 2\, e^0 \cdot 0 + c$

$\boxed{1 = c}$

eqn ① become

$y \cdot e^{-x^2} = \int 2\, e^{-x^2}\, dx + 1$

$y = e^{x^2}\left[\int 2\, e^{-x^2}\, dx + 1\right] \qquad ②$

$\int e^{-x^2}\, dx = \int_0^x e^{-u^2}\, du \qquad\qquad ③$

we know $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\, du$

$$\therefore 2 \int_0^x e^{-u^2} du = \sqrt{\pi} \ erf(x) \quad \text{---} \ \text{④}$$

hence by eq^n ②, ⑨ and ④

$$\boxed{y = e^{x^2} \left[1 + \sqrt{\pi} \ erf(x) \right]}$$

is required solution.

Q. 5 b) Given that $L\{f(t); p\} = F(p)$

Show that $\int\limits_0^\infty \dfrac{f(t)}{t} dt = \int\limits_0^\infty F(x) dx.$

Hence evaluate the integral $\int\limits_0^\infty \dfrac{e^{-t} - e^{-3t}}{t} dt.$

Sol^n: i) Given that $L\{f(t); p\} = F(p)$

We have,

$$L\left\{\dfrac{f(t)}{t}\right\} = \int\limits_0^\infty e^{-st} \dfrac{f(t)}{t} dt$$

$$= \int\limits_s^\infty f(s)\, ds$$

$$= \int\limits_s^\infty F(x)\, dx$$

$$= \int\limits_s^0 F(x)\, dx + \int\limits_0^\infty F(x)\, dx$$

$$= -\int\limits_0^s F(x)\, dx + \int\limits_0^\infty F(x)\, dx \quad -①$$

Taking limits of both sides of ① as $s \to 0^+$ and assuming that the integral converges, we get

$$\boxed{\int\limits_0^\infty \dfrac{f(t)}{t} dt = \int\limits_0^\infty F(x)\, dx}$$

Hence proved.

ii) $\int\limits_{0}^{\infty} \dfrac{e^{-t} - e^{-3t}}{t}\, dt$

$\Rightarrow L\left\{e^{-t} - e^{-3t}\right\}$

$= L\left\{e^{-t}\right\} - L\left\{e^{-3t}\right\}$

$= \dfrac{1}{s+1} - \dfrac{1}{s+3} = f(s),\ \text{say} \quad - \text{①}$

$L\left\{\dfrac{e^{-t} - e^{-3t}}{t}\right\} = \int\limits_{s}^{\infty} f(s)\, ds$

$= \int\limits_{s}^{\infty}\left(\dfrac{1}{s+1} - \dfrac{1}{s+3}\right) ds,\ \text{using ①}$

$= \Big[\log(s+1) - \log(s+3)\Big]_{s}^{\infty}$

$= \left[\log\dfrac{(s+1)}{(s+3)}\right]_{s}^{\infty}$

$= \lim\limits_{s\to\infty}\log\dfrac{s+1}{s+3} - \log\left(\dfrac{s+1}{s+3}\right)$

$= \lim\limits_{s\to\infty}\log\dfrac{1 + 1/s}{1 + 3/s} + \log\left(\dfrac{s+3}{s+1}\right)$

$= \log(1) + \log\left(\dfrac{s+3}{s+1}\right)$

$= \log\left(\dfrac{s+3}{s+1}\right)$

or $\int\limits_{0}^{\infty} e^{-st}\left(\dfrac{e^{-t} - e^{-3t}}{t}\right) dt = \log\left(\dfrac{s+3}{s+1}\right) - \text{②}$

- by definition of Laplace Transform

Taking limit of both sides of ②

as $s \to 0$, we get

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} \, dt = \log\left(\frac{3}{1}\right) = \log 3$$

Q.5 c) A cylinder of radius 'a' touches a vertical wall along a generating line. Axis of the cylinder is fixed horizontally. A uniform flat beam of length 'l' and weight 'w' rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. If frictional forces are neglected then show that

$$\frac{a}{l} = \frac{\sqrt[4]{5} + 5}{4\sqrt{2}}$$

Also, find the reactions of the cylinder and wall.

sol^n ⟹

A Given the rod AB is in the position of ~~equation~~ equilibrium on the cylinder and against the wall under the forces then it must be true below.

fig. 01

i) Reaction R ⊥ wall at B

ii) Reaction S at A passing through the centre D.

iii) weight W acting at the middle point G of the rod vertically downwards
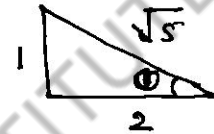
$\angle ABL = \angle OGB = 45°$ (Given)

Let $\angle AOG = \theta$

Applying m:n theorem in $\triangle AOB$

$(1+1) \cot 45° = 1 \cot\theta - 1 \cot 90°$

$2 = \cot\theta$

$\cot\theta = 2$ ——— ①

by applying Lami's theorem

$$\frac{R}{\sin(180-\theta)} = \frac{S}{\sin 90} = \frac{W}{\sin(90+\theta)}$$

$$\frac{R}{\sin\theta} = \frac{S}{1} = \frac{W}{\cos\theta}$$

$R = W \tan\theta = \frac{W}{2}$ since eq^n ①

Reaction on the wall $R = \frac{W}{2}$

and $S = W \sec\theta = W\sqrt{1+\tan^2\theta}$

$= W\sqrt{1+1/4} = \frac{\sqrt{5}}{2} W$

$DL = DM + ML = DM + AN = a\sin\theta + l\sin 45°$

$\Rightarrow a = a\left(\frac{1}{\sqrt{5}}\right) + l\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \frac{l}{\sqrt{2}} = \left(\frac{\sqrt{5}-1}{\sqrt{5}}\right) a$

$$\Rightarrow \quad \frac{a}{l} = \frac{\sqrt{s} \cdot 1}{\sqrt{2} \; \sqrt{s-1}} = \frac{\sqrt{s}}{\sqrt{2}} \cdot \frac{\sqrt{s+1}}{\sqrt{s-1} \; \sqrt{s+1}}$$

$$= \frac{\sqrt{s}}{\sqrt{2}} \cdot \frac{\sqrt{s}+1}{4} = \frac{\sqrt{s} + s}{4\sqrt{2}}$$

$$\therefore \quad \boxed{\frac{a}{l} = \frac{s + \sqrt{s}}{4\sqrt{2}}} \qquad \text{Hence proved}$$

similar, Reaction on the wall

$$R = w \tan\alpha = w \left(\frac{1}{2}\right) = \frac{w}{2}$$
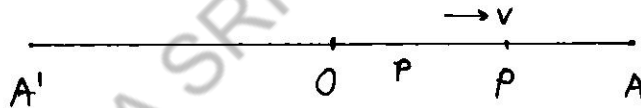
$$\therefore \quad \boxed{R = \frac{w}{2}}$$

is required solution.

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Topic

Remarks

AMANASRI
Learning on time
IAS/IFoS Maths Optional

Q. 5 d) A particle is moving under Simple Harmonic Motion of period T about a centre O. It passes through the point P with velocity v along the direction OP and OP = P. Find the time that elapses before the particle returns to the point P. What will be the value of ρ when the elapsed time is $\frac{T}{2}$?

Sol^n: Let the equation of the S.H.M. with centre O as origin be $\frac{d^2x}{dt^2} = -\mu x$

The time period $T = \frac{2\pi}{\sqrt{\mu}}$

Let the amplitude be a.

Then $\left(\frac{dx}{dt}\right)^2 = \mu (a^2 - x^2)$ — ①

when the particle passes through P its velocity is given to be 'V' in the direction OP. Also OP = P. So putting

$x = P$ and $\frac{dx}{dt} = V$ in ①, we get

$$v^2 = \mu (a^2 - \rho^2). \quad - ②$$

Let A be an extremity of the motion. From P the particle comes to instantaneous rest at A and then returns back to P. In S.H.M. the time from P to A is equal to the time from A to P.

∴ the required time $= 2 \times$ (time from A to P)

Now for the motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu(a^2-x^2)} \quad \text{or} \quad dt = -\frac{dx}{\sqrt{\mu(a^2-x^2)}} \quad - \text{③}$$

Let t_1 be the time from A to P.

Then at A, $t=0$, $x=a$ and at P, $t=t_1$ & $x=P$.

Therefore integrating ③, we get

$$\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}} \int_a^P \frac{-dx}{\sqrt{a^2-x^2}}$$

$$t_1 = \frac{1}{\sqrt{\mu}} \left[\cos^{-1} \frac{x}{a} \right]_a^P$$

∴ $t_1 = \frac{1}{\sqrt{\mu}} \left[\cos^{-1}\frac{P}{a} - \cos^{-1} 1 \right]$

∴ $t_1 = \frac{1}{\sqrt{\mu}} \cos^{-1} \frac{P}{a}$

Hence the required time

$$= 2 t_1$$

$$= \frac{2}{\sqrt{\mu}} \cos^{-1} \frac{P}{a}$$

$$= \frac{2}{\sqrt{\mu}} \tan^{-1} \left(\frac{\sqrt{a^2-P^2}}{P} \right)$$

$$= \frac{2}{\sqrt{\mu}} \tan^{-1}\left(\frac{v}{\rho\sqrt{\mu}}\right)$$

$$\left[\because \text{from } ②, \sqrt{a^2-\rho^2} = \frac{v}{\sqrt{\mu}}\right]$$

$$= \frac{2}{2\pi/T} \tan^{-1}\left(\frac{v}{\rho(2\pi/T)}\right)$$

$$\left(\because T = \frac{2\pi}{\sqrt{\mu}} \quad \therefore \sqrt{\mu} = \frac{2\pi}{T}\right)$$

$$= \frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi\rho}\right)$$

$\therefore$
$$\boxed{\text{Time elapsed before particle returns to point } \rho = \frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi\rho}\right)}$$

To find value of ρ when elapsed time is $\frac{T}{2}$

$$\therefore \frac{T}{2} = \frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi\rho}\right)$$

$$\therefore \tan\frac{\pi}{2} = \frac{vT}{2\pi\rho}$$

$$\infty = \frac{vT}{2\pi\rho}$$

$$\Rightarrow \boxed{\rho = 0}$$

Q. 5 e) If
$$\bar{a} = \sin\theta\, i + \cos\theta\, j + \theta\, k$$
$$\bar{b} = \cos\theta\, i - \sin\theta\, j - 3k$$
$$\bar{c} = 2i + 3j - 3k$$
then find the values of the derivative of the vector function $\bar{a} \times (\bar{b} \times \bar{c})$ w.r.t. θ at $\theta = \dfrac{\pi}{2}$ and $\theta = \pi$

Sol^n

$$\bar{b} \times \bar{c} = \begin{vmatrix} i & j & k \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix}$$

$$= i\,(3\sin\theta + 9) + j\,(-6 + 3\cos\theta)$$
$$+ k\,(3\cos\theta + 2\sin\theta)$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} i & j & k \\ \sin\theta & \cos\theta & \theta \\ 3\sin\theta + 9 & -6 + 3\cos\theta & 3\cos\theta + 2\sin\theta \end{vmatrix}$$

$$= i\,(3\cos^2\theta + 2\sin\theta\cos\theta + 6\theta - 3\theta\cos\theta)$$
$$+ j\,(3\theta\sin\theta + 9\theta - 3\sin\theta\cos\theta - 2\sin^2\theta)$$
$$+ k\,(-6\sin\theta + 3\sin\theta\cos\theta - 3\sin\theta\cos\theta - 9\cos\theta)$$

$$= i\,(3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta)$$
$$+ j\,(-2\sin^2\theta - 3\sin\theta\cos\theta + 3\theta\sin\theta + 9\theta)$$
$$+ k\,(-6\sin\theta - 9\cos\theta)$$

$$\frac{d}{d\theta}\left[\bar{a} \times (\bar{b} \times \bar{c})\right] = \left[-6\sin\theta\cos\theta + 2\cos2\theta - 3\cos\theta\right.$$
$$\left. + 3\theta\sin\theta + 6\right] i$$
$$+ j\left[-3(-\sin^2\theta + \cos^2\theta) - 4\sin\theta\cos\theta\right.$$
$$\left. + 3(\theta\cos\theta + \sin\theta) + 9\right]$$
$$+ k\left[-6\cos\theta + 9\sin\theta\right]$$

i) At $\theta = \dfrac{\pi}{2}$, $\sin \dfrac{\pi}{2} = 1$ and $\cos \dfrac{\pi}{2} = 0$

$\therefore \dfrac{d}{d\theta} [\bar{a} \times (\bar{b} \times \bar{c})] = \left(0 + (-2) + \dfrac{3\pi}{2} + 6 \right) i$

$\qquad\qquad\qquad + \left(3 + 0 + 3 + 9 \right) j$

$\qquad\qquad\qquad + \left(-6 \times 0 + 9 \right) k$

$\qquad\qquad = \left(\dfrac{3\pi}{2} + 4 \right) i + 15 j + 9k$

$\therefore$ at $\theta = \dfrac{\pi}{2}$,

$$\boxed{\dfrac{d}{d\theta} [\bar{a} \times (\bar{b} \times \bar{c})] = \left(\dfrac{3\pi + 8}{2} \right) i + 15 j + 9k}$$

ii) At $\theta = \pi$, $\sin \pi = 0$ and $\cos \pi = -1$

$\therefore \dfrac{d}{d\theta} [\bar{a} \times (\bar{b} \times \bar{c})] = \left(0 + 2 + 3 + 6 \right) i$

$\qquad\qquad\qquad + \left(-3 - 3\pi + 9 \right) j$

$\qquad\qquad\qquad + \left(6 + 0 \right) k$

$\qquad\qquad = 11 i + (6 - 3\pi) j + 6k$

$\therefore$ at $\theta = \pi$

$$\boxed{\dfrac{d}{d\theta} [\bar{a} \times (\bar{b} \times \bar{c})] = 11 i + (6 - 3\pi) j + 6k}$$

Q. 6 a) Solve the differential equation:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$$

Soln: Given $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$, —①

where $D \equiv d/dx$

Its auxiliary equation is

$$D^3 - 3D^2 + 4D - 2 = 0$$

or $D^2(D-1) - 2D(D-1) + 2(D-1) = 0$

or $(D-1)(D^2 - 2D + 2) = 0$

giving $D = 1, \left(\dfrac{2 \pm \sqrt{4-8}}{2}\right)$

i.e. $D = 1, 1 \pm i$

$$\therefore \boxed{C.F. = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x),}$$

c_1, c_2, c_3 being arbitrary constants

P.I. corresponding to e^x

$$= \frac{1}{D^3 - 3D^2 + 4D - 2} e^x$$

$$= \frac{1}{(D-1)(D^2 - 2D + 2)} e^x$$

$$= \frac{1}{D-1} \cdot \frac{1}{1-2+2} e^x$$

$$= \frac{1}{D-1} e^x \cdot 1$$

$$= e^x \frac{1}{(D+1)-1} \cdot 1$$

$$= e^x \frac{1}{D} \cdot 1$$

$$= x e^x$$

P. I. corresponding to $\cos x$

$$= \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x$$

$$= \frac{1}{D^2 \cdot D - 3D^2 + 4D - 2} \cos x$$

$$= \frac{1}{(-1^2)D - 3(-1^2) + 4D - 2} \cos x$$

$$= \frac{1}{3D+1} \cos x = (3D-1) \frac{1}{9D^2 - 1} \cos x$$

$$= (3D-1) \frac{1}{9(-1^2)-1} \cos x$$

$$= \frac{-1}{10} (3D\cos x - \cos x)$$

$$= \frac{-1}{10} (-3\sin x - \cos x) = \frac{3\sin x + \cos x}{10}$$
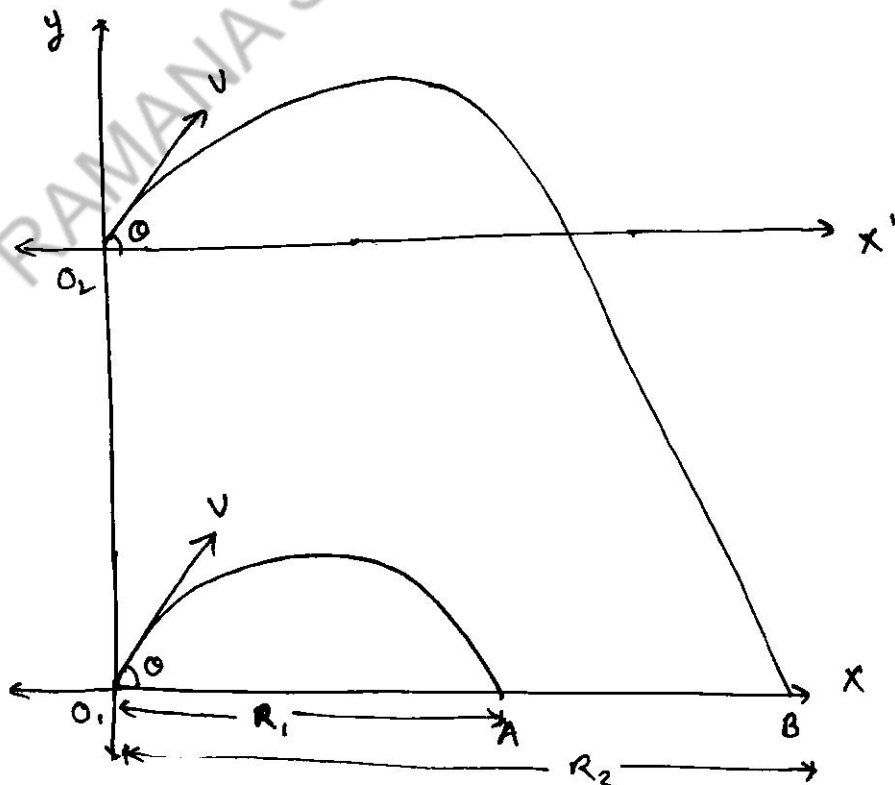
$\therefore$ Required solution is

$$\boxed{\begin{array}{l} y = e^x (C_1 + C_2 \cos x + C_3 \sin x) + x e^x \\ \qquad + \frac{(3\sin x + \cos x)}{10} \end{array}}$$

Q.6 b) when a particle is projected from a point O_1 on the sea level with a velocity V and angle of projection θ with the ~~horizontal~~ horizon in a vertical plane, its horizontal range is R_1. If it is further projected from a point O_2, which is vertically above O_1 at a height h in the same vertical plane, with the same velocity V and same angle θ with the horizon, its horizontal range is R_2. Prove that $R_2 > R_1$ and $(R_2 - R_1) : R_1$ is equal to

$$\frac{1}{2}\left\{\sqrt{\left(1 + \frac{2gh}{v^2 \sin^2\theta}\right)} - 1\right\} : 1$$

Sol^n ⟹

let R_1 be the original range. then

$$R_1 = \frac{2v^2 \sin \theta \cos \theta}{g}$$

let o be a point a height h above the water level. let R_2 be the range on the sea when the shot is fired from O.

Referred to the horizontal and vertical lines OX & OY in the plane of projection as the co-ordinates axes of the points where the shot strikes the water are $(R_1, -h)$ (R_2, h)

the point $R_2 - R_1$ lies on the curve

$$y = x \tan \theta - \frac{1}{2} \frac{g x^2}{v^2 \cos^2 \theta}$$

$$-h = R_2 \tan \theta - \frac{1}{2} \frac{g R_2^2}{v^2 \cos^2 \theta}$$

or $R_2^2 - \frac{2}{g} v^2 \sin \theta \cos \theta R_2 - \frac{2}{g} v^2 h \cos^2 \theta = 0$

$$R_2^2 - R_2 R_1 - \frac{2}{g} v^2 h \cos^2 \theta = 0$$

$$R_2^2 - R_2 R_1 = \frac{2}{g} v^2 h \cos^2 \theta$$

$$\left(R_2 - \frac{1}{2} R_1\right)^2 = \frac{1}{4} R_2^2 + \frac{2}{g} v^2 h \cos^2 \theta =$$

$$= \frac{R_1^2}{4}\left[1 + \frac{1}{R_1^2} \cdot \frac{8}{9} v^2 h \cos^2\theta\right]$$

$$\therefore \left(R_2 - \frac{1}{2}R_1\right)^2 = \frac{R_1^2}{4}\left[1 + \frac{9}{4v^2\sin^2\alpha\cos^2\theta} \cdot \frac{8}{9} v^2 h \cos^2\theta\right]$$

$$[\text{by } (1)]$$

$$= \frac{R_1^2}{4}\left[1 + \frac{2gh}{v^2\sin^2\theta}\right]$$

$$R_2 - \frac{1}{2}R_1 = \frac{1}{2}R_1\left[1 + \frac{2gh}{v^2\sin^2\theta}\right]^{1/2}$$

so that:

$$R_2 - R_1 = \frac{1}{2}R_1\left(1 + \frac{2gh}{v^2\sin^2\theta}\right)^{1/2} - \frac{1}{2}R$$

$$\frac{R_2 - R_1}{R_1} = \frac{1}{2}\left[\left(1 + \frac{2gh}{v^2\sin^2\theta}\right)^{1/2} - 1\right]$$

hence we proved required result.

g. 5 (c) Evaluate the integral $\iint\limits_{S} (3y^2z^2\hat{i} + 4z^2x^2\hat{j} + z^2y^2\hat{k}).\hat{n}\,ds$

where S is the upper part of the surface $4x^2 + 4y^2 + 4z^2 = 1$ above the plane $z = 0$ and bounded by the xy-plane. Hence verify Gauss divergence theorem.

soln $\rightarrow$ by divergence theorem,

$\iint\limits_{S} (3y^2z^2 i + 4z^2x^2 j + z^2y^2 \hat{k}).\hat{n}\,ds$

$= \iiint\limits_{V} \text{div} (3y^2z^2 i + 4z^2x^2 j + z^2y^2 k)\,dv$

where V is the volume enclosed by S.

$= \iiint\limits_{V} \left[\frac{\partial}{\partial x}(3y^2z^2) + \frac{\partial}{\partial y}(4z^2x^2) + \frac{\partial}{\partial z}(z^2y^2) \right] dv$

$= \iiint\limits_{V} 2zy^2 \, dv = 2\iiint\limits_{V} zy^2 dv$

we shall use spherical polar co-ordinate (r, θ, ϕ) to evaluate this triple integral.

In polar,

$dv = dr, \; rd\theta \cdot r\sin\theta \, d\phi$

$= r^2 \sin\theta \, dr\,d\theta\,d\phi.$

~~A51~~ · ~~A51~~ Also $z = r\cos\theta$, $y = r\sin\theta \sin\phi$

To. cover V the limits of r will be 0 to $\frac{1}{2}$, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

The triple integral is

$$= 2 \int_{r=0}^{1/2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r\cos\theta \, r^2 \sin^2\theta \sin^2\phi \times r^2 \sin\theta \, dr \, d\theta \, d\phi.$$

$$= 2 \int_{0}^{1/2} \int_{0}^{\pi/2} \int_{0}^{2\pi} r^5 \sin^3\theta \cos\theta \sin^2\phi \, dr \, d\theta \, d\phi$$

$$= 2 \left(\frac{r^6}{6}\right)_{0}^{1/2} \cdot \int_{0}^{\pi/2} \int_{0}^{2\pi} \cos\theta \sin^3\theta \sin^2\phi \, d\theta \, d\phi$$

$$= 2 \times \frac{1}{6} \times \frac{1}{2^6} \times \frac{2}{4 \cdot 2} \int_{0}^{2\pi} \sin^2\phi \, d\phi$$

$$= \frac{1}{768} \times 4 \times \int_{0}^{\pi/2} \sin^2\phi \, d\phi$$
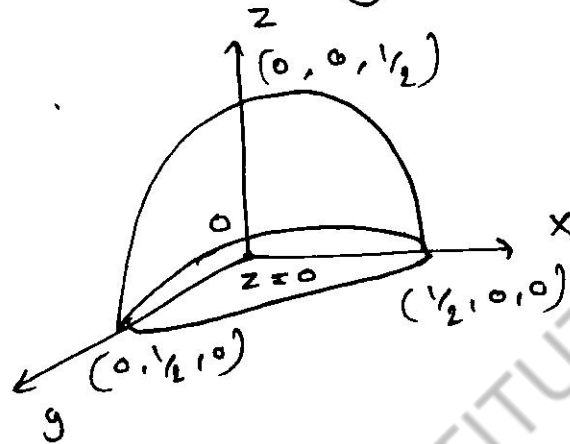
$$= \frac{1}{768} \times 4 \times \frac{1}{2} \times \frac{\pi}{4}$$

$$= \frac{\pi}{768}$$

$$\boxed{V = \frac{\pi}{768}}$$ ————————①

verification of Gauss divergence —

Given —

Given $\phi \equiv 4x^2 + 4y^2 + 4z^2 - 1 = 0$

let $\bar{F} = 3y^2z^2 i + 4z^2x^2 j + z^3y^2 k$

we have. $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = 2(xi + yj + zk)$

$F.\hat{n} = 2(3xy^2z^2 + 4x^2z^2y + z^3y^2)$

we have $ds = \dfrac{dx\,dy}{|\hat{n}.k|} = \dfrac{dx\,dy}{2z}$

$$\iint\limits_{S} \bar{F}.\bar{n}\,ds = \iint\limits_{R} 2\dfrac{(3x^2y^2z^2 + 4x^2yz^2 + y^2z^3)}{2z}\,dx\,dy$$

$$= \iint\limits_{R} (3xy^2z + 4x^2yz + y^2z^2)\,dx\,dy \quad\text{——①}$$

corresponding to $x^2 + y^2 = \dfrac{1}{4}$

$x = r\cos\theta \quad ; \quad r = r\sin\theta$

$\therefore dx\,dy = r\,dr\,d\theta, \quad 0 \le x \le \dfrac{1}{2}$

$$0 \le \theta \le 2\pi$$

$\because \quad x^2 + y^2 + z^2 = \frac{1}{4} \quad \Rightarrow \quad z^2 = \frac{1}{4} - x^2 - y^2$

$$\Rightarrow \quad z = \sqrt{\frac{1}{4} - r^2}$$

$\therefore \iint\limits_{S} \bar{F} \cdot \hat{n} \, ds = \int\limits_{r=0}^{1/2} \int\limits_{\theta=0}^{2\pi} \Big[3r^3 \cos\theta \, \sin^2\theta$

$$\times \sqrt{\tfrac{1}{4} - r^2} + 4r^3 \cos^2\theta \, \sin\theta \sqrt{\tfrac{1}{4} - r^2}$$

$$+ \; r^2 \sin^2\theta \left(\tfrac{1}{4} - r^2 \right) \Big] r \, dr \, d\theta$$

$= \int\limits_{r=0}^{1/2} \int\limits_{0}^{2\pi} \Big[r^4 \sqrt{\tfrac{1}{4} - r^2} \left(3\cos\theta \sin^2\theta + 4\cos^2\theta \sin\theta \right)$

$$+ \; \frac{r^3 \sin^2\theta}{4} - r^5 \sin^2\theta \Big] \, dr \, d\theta$$

$= \int\limits_{0}^{2\pi} \Big[\sin^2\theta \left(\frac{1}{16 \times 2^4} - \frac{1}{2 \times 6} \right) \Big] \, d\theta$

$$+ \int\limits_{r=0}^{1/2} r^4 \sqrt{\tfrac{1}{4} - r^2} \left(\int\limits_{\theta=0}^{2\pi} 3\cos\theta \sin^2\theta + 4\cos^2\theta \cdot \sin\theta \right) d\theta$$

$$= \frac{\pi}{768} \qquad \underline{\hspace{2cm}} \textcircled{2}$$

by ① & ② we verified Gauss

divergence theorem.

Q. 7 a) i) Find the solution of the differential equation: $\dfrac{dy}{dx} = -\dfrac{2xy^3 + 2}{3x^2 y^2 + 8e^{4y}}$

Sol^n:

$$\frac{dy}{dx} = -\frac{2xy^3 + 2}{3x^2 y^2 + 8e^{4y}}$$

$$(2xy^3 + 2)\,dx + (3x^2 y^2 + 8e^{4y})\,dy = 0$$

$$M\,dx + N\,dy = 0$$

$\therefore$ $M = 2xy^3 + 2$ $N = 3x^2 y^2 + 8e^{4y}$

$$\frac{\partial M}{\partial y} = 6xy^2 \quad ; \quad \frac{\partial N}{\partial x} = 6xy^2$$

$$\therefore \frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x}$$

Hence above ODE is exact.

Therefore solution is

$$\int_{y=c} M\,dx + \int_{\text{no } x \text{ term}} N\,dy = c$$

$$\int_{y=c} (2xy^3 + 2)\,dx + \int 8e^{4y}\,dy = c$$

$$\Rightarrow 2y^3 \frac{x^2}{2} + 2x + 8\frac{e^{4y}}{4} = c$$

$$\boxed{x^2 y^3 + 2x + 2e^{4y} = c}$$

which is the required solution.

Q. 7 a) ii) Reduce the equation $x^2 p^2 + y(2x+y)p + y^2 = 0$ to Clairaut's form by the substitution $y = u$ and $xy = v$. Hence solve the equation and show that $y + 4x = 0$ is a singular solution of the differential equation.

Sol^n:

Given equation is $x^2 p^2 + yp(2x+y) + y^2 = 0$ — ①

Given $y = u$ and $xy = v$. — ②

Differentiating ②

$dy = du$ and $x\,dy + y\,dx = dv$

$\therefore \dfrac{x\,dy + y\,dx}{dy} = \dfrac{dv}{du}$

or $x + y\dfrac{dx}{dy} = \dfrac{dv}{du}$

or $x + \dfrac{y}{P} = P$

or $\dfrac{y}{P} = P - x$

or $p = \dfrac{y}{(P-x)}$ where $p = \dfrac{dy}{dx}$, $P = \dfrac{dv}{du}$

putting $p = y/(P-x)$ in ①, we have

$\dfrac{x^2 y^2}{(P-x)^2} + \dfrac{y^2}{P-x}(2x+y) + y^2 = 0$

or $x^2 + (P-x)(2x+y) + (P-x)^2 = 0$

or $\quad Py - xy + P^2 = 0$

or $\quad \boxed{v = uP + P^2}$ using ② $\qquad - ③$

③ is in Clairaut's form.

So replacing P by c its general solution is

$$v = uc + c^2$$

or $\quad xy = yc + c^2$, c being an arbitrary constant.

or $\quad c^2 + cy - xy = 0$

which is a quadratic equation in c and hence c-discriminant relation is

$$y^2 - 4 \cdot 1 \cdot (-xy) = 0$$
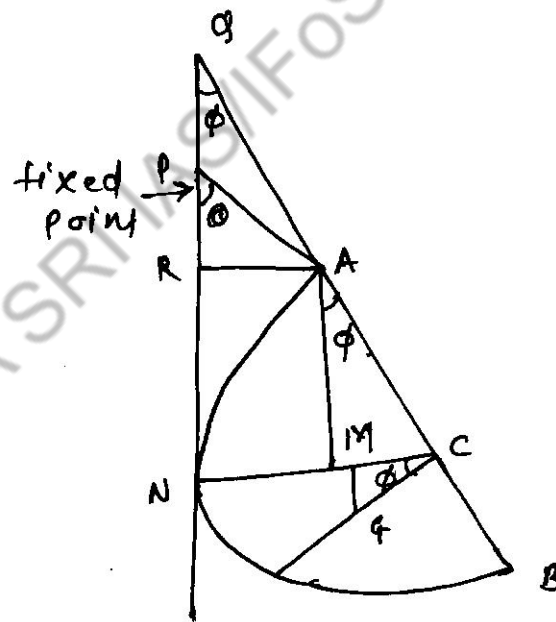
or $\quad y(y + 4x) = 0$

Since $y = 0$ and $\underline{y + 4x = 0}$ both satisfy ①, so these are both <u>singular solutions</u>.

9.7 (b) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface is in contact. If Θ is the angle of inclination of the string with vertical and ϕ is the angle of inclination of the plane base of the hemisphere to the vertical, then find the value of $(\tan\phi - \tan\Theta)$.

$sol^n \Rightarrow$

let the length of the string $AP = l$
and let the radius of the hemisphere be a so that $\ell G = \dfrac{3a}{8}$

P is a fixed point in the wall. Give a small displacement to the system such that Θ and ϕ become $\Theta + \delta\Theta$ and $\phi + \delta\phi$ respectively.

Equation of virtual work is

$$w \delta (\text{depth of } G \text{ below } p) = 0$$

or $\delta (\text{depth of } G \text{ below } p) = 0$

$$\delta (PR + RN + MG) = 0$$

or $\delta (l \cos\theta + a \cos\phi + \frac{3a}{8} \sin\phi) = 0$

or $-l \sin\theta \, d\theta + \delta\phi (-a \sin\phi +$

$$\frac{3a}{8} \cos\phi) = 0 \quad \text{------} \quad (1)$$

from the figure it is clear that

$$CN = CL + LN = a \sin\phi + l \sin\theta$$

$$a = a \sin\phi + l \sin\theta$$

taking differential $a \cos\phi . \delta\phi +$

$$l \cos\theta \, \delta\theta = 0$$

$$\frac{\delta\theta}{\delta\phi} = -\frac{a}{l} \frac{\cos\phi}{\cos\theta}$$

using this in (1) we get

$$-l \sin\theta . \left(-\frac{a \cos\phi}{l \cos\theta}\right) +$$

$$\left(-a \sin\phi + \frac{3a}{8} \cos\theta\right) = 0$$

$$\tan\theta \cdot \cos\phi - \sin\phi + \frac{3}{8}\cos\phi = 0$$

dividing by $\cos\phi$,

$$\tan\theta - \tan\phi + \frac{3}{8} = 0$$

$$\boxed{\tan\phi - \tan\theta = \frac{3}{8}}$$

Hence required solution.

Q. 7 c) If the tangent to a curve makes a constant angle θ with a fixed line, then prove that the ratio of radius of torsion to radius of curvature is proportional to $\tan\theta$. Further prove that if this ratio is constant, then the tangent makes a constant angle with a fixed direction.

Sol^n: Let's consider a curve in 3-D space parametrized by arc length, denoted by $\vec{r}(s)$, where s is the arc length parameter. The radius of curvature R is given by:

$$R = \left| \frac{d\vec{T}}{ds} \right|^{-1}$$

where $\vec{T}$ is the unit tangent vector to the curve.

Now, let's consider the tangent vector $\vec{T}(s)$ at a point on the curve. According to the problem, the tangent makes a constant angle θ with a fixed line. This means that $\vec{T}(s)$ is rotating about a fixed axis with a constant angular velocity ω, such that

$$\theta = \omega s$$

Differentiating $\vec{T}(s)$ with respect to s gives us $\dfrac{d\vec{T}}{ds} = w\vec{N}$, where $\vec{N}$ is the unit normal vector.

Now, the radius of torsion T is defined as the reciprocal of the rate of change of Θ w.r.t. arc length s:

$$T = \left|\frac{d\Theta}{ds}\right|^{-1} = \left|\frac{d(ws)}{ds}\right|^{-1} = |w|^{-1}$$

Now, the radiu

So, we have found expressions for both the radius of curvature and the radius of torsion.

Now, let's examine the relationship between T and R:

$$\frac{T}{R} = |w|^{-1} \cdot \left|\frac{d\vec{T}}{ds}\right|^{-1}$$

Since $\left|\dfrac{d\vec{T}}{ds}\right| = |w\vec{N}| = |w| \cdot |\vec{N}| = |w|$,

we can simplify the expression:

$$\boxed{\frac{T}{R} = |w|^{-1} \cdot |w|^{-1} = |w|^{-2} = \tan^2(\Theta)}$$

So, we have proved that the ratio of the radius of torsion to radius of curvature is proportional to $\tan^2(\Theta)$.

Now, let's prove the second part of the statement.

If this ratio $\frac{I}{R}$ is constant, it implies that $\tan^2(\theta)$ is constant. Since $\tan^2(\theta)$ is constant, θ must be constant as well, because the tangent of a constant angle is also a constant. Therefore, if the ratio $\frac{I}{R}$ is constant, the tangent makes a constant angle θ with a fixed direction.

Q. 8 a) Solve the following initial value problem by using Laplace transform technique:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y(t) = f(t),$$

$y(0) = 1$, $y'(0) = 0$ and $f(t)$ is a given function of t.

Sol^n: Given $y'' - 4y' + 3y = f(t)$ — ①

where $y'' = \dfrac{d^2y}{dt^2}$, $y' = \dfrac{dy}{dt}$

with initial condition

$y(0) = 1$ and $y'(0) = 0$ — ②

Taking Laplace transform of both sides of ①, we get

$$L\{y''\} - 4L\{y'\} + 3L\{y\} = L\{f(t)\}$$

or $s^2 L\{y\} - sy(0) - y'(0) - 4[sL\{y\} - y(0)]$

$\qquad + 3L(y) = f(s)$ — ③

where $L\{f(t)\} = f(s)$ so that $L^{-1}\{f(s)\} = f(t)$

— ④

Using ②,

③ $\Rightarrow (s^2 - 4s + 3)L\{y\} - s + 4 = f(s)$

or $L\{y\} = \dfrac{s - 4 + f(s)}{(s^2 - 4s + 3)}$

$\qquad = \dfrac{s - 4}{(s-1)(s-3)} + \dfrac{f(s)}{(s-1)(s-3)}$

or $L\{y\} = \frac{1}{2}\left[\frac{3}{s-1} - \frac{1}{s-3}\right] + \frac{1}{2}f(s)\left[\frac{1}{s-3} - \frac{1}{s-1}\right]$

on resolving into partial fractions

$\therefore y = \frac{3}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2}L^{-1}\left\{f(s)\frac{1}{s-3}\right\}$

$\qquad - \frac{1}{2}L^{-1}\left\{f(s)\frac{1}{s-1}\right\}$

$y = \frac{3}{2}e^{t} - \frac{1}{2}e^{3t} + \frac{1}{2}L^{-1}\{f(s)g(s)\}$

$\qquad - \frac{1}{2}L^{-1}\{f(s)h(s)\} \qquad -⑤$

where $g(s) = \frac{1}{s-3}$ and $h(s) = \frac{1}{s-1} \qquad -⑥$

so that $g(t) = L^{-1}\{g(s)\} = e^{3t}$
and $h(t) = L^{-1}\{h(s)\} = e^{t}$ $\qquad\Big\}⑦$

Now, by using the convulation theorem and ⑦, we have

$L^{-1}\{f(s)g(s)\} = \int_{0}^{t} f(u)g(t-u)\,du$

$\qquad = \int_{0}^{t} f(u)e^{3(t-u)}\,du$

$\qquad = e^{3t}\int_{0}^{t} f(u)e^{-3u}\,du \qquad -⑧$

and $L^{-1}\{f(s)h(s)\} = \int_{0}^{t} f(u)h(t-u)\,du$

$\qquad = \int_{0}^{t} f(u)e^{t-u}\,du$

$\qquad = e^{t}\int_{0}^{t} f(u)e^{-u}\,du \qquad -⑨$

Using ⑧ and ⑨, ⑤ reduces to

$$y = \frac{1}{2}(3e^t - e^{3t}) + \frac{1}{2}e^{3t}\int_0^t f(u)e^{-3u}\,du$$

$$- \frac{1}{2}e^t\int_0^t f(u)e^{-u}\,du$$

Q. 8. b) A particle is projected from an apse at a distance $\sqrt{c}$ from the centre of force with a velocity $\sqrt{\dfrac{2\lambda c^3}{3}}$ and is moving with central acceleration $\lambda(r^5 - c^2 r)$. Find the path of motion of this particle. Will that be the curve $x^4 + y^4 = c^2$?

Sol^n: Here the central acceleration

$$P = \lambda(r^5 - c^2 r) = \lambda\left(\frac{1}{u^5} - \frac{c^2}{u}\right)$$

$\therefore$ The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2}$$

$$= \frac{\lambda}{u^2}\left(\frac{1}{u^5} - \frac{c^2}{u}\right)$$

$$= \lambda\left(\frac{1}{u^7} - \frac{c^2}{u^3}\right)$$

Multiplying both sides by $2\left(\dfrac{du}{d\theta}\right)$ and then integrating, we have

$$v^2 = h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \lambda\left[\frac{-1}{3u^6} + \frac{c^2}{u^2}\right] + A \quad —①$$

where A is constant.

But initially, when $r = \sqrt{c}$ ie. $u = \dfrac{1}{\sqrt{c}}$

$$\frac{du}{d\theta} = 0 \quad \text{(at an apse)} \quad \text{and}$$

$$v = \sqrt{\frac{2\lambda c^3}{3}}$$

$\therefore$ from ①, we have

$$\left(\sqrt{\frac{2\lambda}{3}c^3}\right)^2 = h^2\left(\frac{1}{\sqrt{c}}\right)^2 = \lambda\left(-\frac{c^3}{3}+c^3\right)+A$$

$$\frac{2\lambda c^3}{3} = \frac{h^2}{c} = \lambda\frac{2c^3}{3}+A$$

$$\therefore \quad A = 0 \quad \text{and} \quad h^2 = \frac{2\lambda}{3}c^4$$

Substituting the values of h^2 and A in ①, we have

$$\frac{2\lambda}{3}c^4\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \lambda\left(\frac{-1}{3u^6}+\frac{c^2}{u^2}\right)$$

$$c^4\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{3}{2}\left(\frac{-1}{3u^6}+\frac{c^2}{u^2}\right)$$

$$c^4\left(\frac{du}{d\theta}\right)^2 = \frac{-1}{2u^6}+\frac{3c^2}{2u^2}-c^4u^2$$

$$= \frac{1}{u^6}\left[\frac{-1}{2}+\frac{3}{2}c^2u^4-c^4u^8\right]$$

$$= \frac{1}{u^6}\left[\frac{-1}{2}-\left(c^4u^8-\frac{3}{2}c^2u^4\right)\right]$$

$$= \frac{1}{u^6}\left[\frac{-1}{2}-\left(c^2u^4-\frac{3}{4}\right)^2+\frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\frac{1}{16}-\left(c^2u^4-\frac{3}{4}\right)^2\right]$$

$$= \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2-\left(c^2u^4-\frac{3}{4}\right)^2\right]$$

$$\therefore \quad c^2u^3\left(\frac{du}{d\theta}\right) = \sqrt{\left(\frac{1}{4}\right)^2-\left(c^2u^4-\frac{3}{4}\right)^2}$$

$$d\theta = \frac{c^2 u^3 \, du}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(c^2 u^4 - \frac{3}{4}\right)^2}} \quad - \textcircled{2}$$

Putting $c^2 u^4 - \frac{3}{4} = z \Rightarrow 4c^2 u^3 \, du = dz$

$$c^2 u^3 \, du = \frac{1}{4} dz$$

$\therefore \textcircled{2} \Rightarrow 4 \, d\theta = \dfrac{dz}{\sqrt{\left(\frac{1}{4}\right)^2 - z^2}}$

Integrating,

$$4\theta + B = \sin^{-1}\left(\frac{z}{1/4}\right) = \sin^{-1}(4z)$$

where B is constant.

$$4\theta + B = \sin^{-1}(4c^2 u^4 - 3)$$

But initially when $u = \frac{1}{\sqrt{c}}$, $\theta = 0$

$\therefore B = \sin^{-1}(1) = \frac{\pi}{2}$

$\therefore 4\theta + \frac{\pi}{2} = \sin^{-1}(4c^2 u^4 - 3)$

$$\sin\left(4\theta + \frac{\pi}{2}\right) = 4c^2 u^4 - 3$$

$$\cos 4\theta = 4c^2 u^4 - 3$$

$$4c^2 u^4 = 3 + \cos 4\theta$$

$$4\frac{c^2}{r^4} = 3 + \cos 4\theta$$

$$4c^2 = r^4\left[3 + (2\cos^2 2\theta - 1)\right]$$

$$4c^2 = 2r^4\left[1 + \cos^2 2\theta\right]$$

$$4c^2 = 2r^4\left[(\cos^2\theta + \sin^2\theta)^2 + (\cos^2\theta - \sin^2\theta)^2\right]$$

$$4c^2 = 4r^4 \left[\cos^4\theta + \sin^4\theta \right]$$

$$c^2 = (r\cos\theta)^4 + (r\sin\theta)^4$$

$$\boxed{c^2 = x^4 + y^4}$$

which is the required equation of the path.

Q. 8. c) For a scalar point function ϕ and vector point function $\vec{f}$, prove the identity $\nabla \cdot (\phi \vec{f}) = \nabla \phi \cdot \vec{f} + \phi(\nabla \cdot \vec{f})$. Also find the value of $\nabla \cdot \left(\dfrac{f(r)}{r} \vec{r} \right)$ and then verify stated identity.

Sol^n: i) We have,

$$\text{div}(\phi \vec{f}) = \nabla \cdot (\phi \vec{f})$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (\phi \vec{f})$$

$$= i \cdot \frac{\partial}{\partial x}(\phi \vec{f}) + j \cdot \frac{\partial}{\partial y}(\phi \vec{f}) + k \cdot \frac{\partial}{\partial z}(\phi \vec{f})$$

$$= \Sigma \left\{ i \cdot \left(\frac{\partial}{\partial x}(\phi \vec{f}) \right) \right\}$$

$$= \Sigma \left\{ i \cdot \left(\frac{\partial \phi}{\partial x} \vec{f} + \phi \frac{\partial \vec{f}}{\partial x} \right) \right\}$$

$$= \Sigma \left\{ i \cdot \left(\frac{\partial \phi}{\partial x} \vec{f} \right) \right\} + \Sigma \left\{ i \cdot \left(\phi \frac{\partial \vec{f}}{\partial x} \right) \right\}$$

$$= \Sigma \left\{ \left(\frac{\partial \phi}{\partial x} i \right) \cdot \vec{f} \right\} + \Sigma \left\{ \phi \left(i \cdot \frac{\partial \vec{f}}{\partial x} \right) \right\}$$

$$\left[\because a \cdot (mb) = (ma) \cdot b = m(a \cdot b) \right]$$

$$= \left\{ \Sigma \frac{\partial \phi}{\partial x} i \right\} \cdot \vec{f} + \phi \Sigma \left(i \cdot \frac{\partial \vec{f}}{\partial x} \right)$$

$$= \underline{\nabla \phi \cdot \vec{f} + \phi(\nabla \cdot \vec{f})}$$

Hence proved

ii) $\nabla \cdot \left(\frac{f(r)}{r} \vec{r} \right)$

$= \nabla \cdot \left\{ \frac{f(r)}{r} (xi + yj + zk) \right\}$

$= \frac{\partial}{\partial x} \left\{ \frac{f(r)}{r} x \right\} + \frac{\partial}{\partial y} \left\{ \frac{f(r)}{r} y \right\} + \frac{\partial}{\partial z} \left\{ \frac{f(r)}{r} z \right\} - \text{①}$

Now $\frac{\partial}{\partial x} \left\{ \frac{f(r)}{r} x \right\} = \frac{f(r)}{r} + x \frac{d}{dr} \left\{ \frac{f(r)}{r} \right\} \frac{\partial r}{\partial x}$

$= \frac{f(r)}{r} + x \left\{ \frac{f'(r)}{r} - \frac{1}{r^2} f(r) \right\} \frac{x}{r}$

$= \frac{f(r)}{r} + \frac{x^2}{r^2} f'(r) - \frac{x^2}{r^3} f(r)$

Similarly $\frac{\partial}{\partial y} \left\{ \frac{f(r)}{r} y \right\} = \frac{f(r)}{r} + \frac{y^2}{r^2} f'(r) - \frac{y^2}{r^3} f(r)$

and $\frac{\partial}{\partial z} \left\{ \frac{f(r)}{r} z \right\} = \frac{f(r)}{r} + \frac{z^2}{r^2} f'(r) - \frac{z^2}{r^3} f(r)$

putting these values in ①, wee get,

$\nabla \cdot \left(\frac{f(r)}{r} \vec{r} \right) = \frac{3}{r} f(r) + \frac{r^2}{r^2} f'(r) - \frac{r^2}{r^3} f(r)$

$= \frac{2}{r} f(r) + f'(r)$

$= \frac{1}{r^2} \left[2r f(r) + r^2 f'(r) \right]$

$= \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right] \qquad - \text{②}$

To verify stated :-

$$\phi = \frac{f(r)}{r} \qquad \vec{f} = \vec{r}$$

$$\nabla \cdot (\phi \vec{f}) = \nabla \cdot \left(\frac{f(r)}{r} \vec{r} \right)$$

$$= \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right] \quad - \text{from } ②$$

i.e. L.H.S $= \nabla \cdot (\phi \vec{f}) = \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right]$

R.H.S. $= \nabla \phi \cdot \vec{f} + \phi (\nabla \cdot \vec{f})$

$$= \nabla \left(\frac{f(r)}{r} \right) \cdot \vec{r} + \frac{f(r)}{r} (\nabla \cdot \vec{r})$$

$$= \left[\nabla f(r) \left(\frac{1}{r} \right) + f(r) \nabla \left(\frac{1}{r} \right) \right] \cdot \vec{r} + 3 \frac{f(r)}{r}$$

$$= \left[\frac{f'(r)}{r} \nabla r + f(r) \left(\frac{-\vec{r}}{r^3} \right) \right] \cdot \vec{r} + 3 \frac{f(r)}{r}$$

$$= \left[\frac{f'(r)}{r^2} \vec{r} - \frac{f(r)}{r^3} \vec{r} \right] \cdot \vec{r} + 3 \frac{f(r)}{r}$$

$$= \left[\frac{f'(r)}{r^2} - \frac{f(r)}{r^3} \right] (\vec{r} \cdot \vec{r}) + 3 \frac{f(r)}{r}$$

$$= \left(\frac{f'(r)}{r^2} - \frac{f(r)}{r^3} \right) r^2 + 3 \frac{f(r)}{r}$$

$$= f'(r) - \frac{f(r)}{r} + 3 \frac{f(r)}{r}$$

$$\approx f'(r) + \frac{2}{r} f(r)$$

$$= \frac{1}{r^2} \left[r^2 f'(r) + 2r\, f(r) \right]$$

$$= \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right]$$

$$\therefore \text{ L.H.S. } = \text{ R.H.S.}$$

Hence stated identity

i.e. $\vec{\nabla} \cdot (\phi \vec{f}) = \vec{\nabla}\phi \cdot \vec{f} + \phi (\vec{\nabla} \cdot \vec{f})$

is verified.