RAMANASRI

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IAS/IFoS Maths Optional

2023 UPSC Maths Optional Paper - 2 Solutions

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UPSC 2023
ESSAY TOPIC
125 MARKS

MATHEMATICS IS THE MUSIC OF REASON

-JAMES JOSEPH SYLVESTER

Q 1. a) Let G be the group of order 10 and G' be a group of order 6. Examine whether there exists a homomorphism of G onto G'. (10 Marks)

Sol^n: Let G be a group of order 10 and G' be another group of order 6.

Let f be a homomorphism from G to G'.

Then by fundamental theorem of group homomorphism (FTGH),

$$G/\ker f \cong f(G), \text{ where } \ker f \leq G \text{ and } f(G) \leq G'$$

And if f is onto then $G/\ker f \cong G'$

Let $f : G \to G'$ be an onto homomorphism

Now order of G is 10 this implies that possible order for $\ker f = 1, 2, 5, 10$

Case I:- If $O(\ker f) = 1$, then

$G/\ker f = G/\{e\} \cong G$ and by FTGH this implies that $G \cong G'$. Which is not possible as $O(G) = 10$ and $O(G') = 6$

Case 2:- If $O(\ker f) = 2$, then
$$O(G/\ker f) = 5$$

By FTGH $G/\ker f \cong G'$ which is again a contradiction as $O(G') = 6$ while $O(G/\ker f) = 5$.

Case 3:- If $O(\ker f) = 5$, then

$$O(G/\ker f) = 2.$$

By FTGH, $G/\ker f \cong G'$ which is a contradiction as above.

Case 4:- If $O(\ker f) = 10$, then this implies the homomorphism is trivial and hence not onto.

This implies that, there does not exist any onto homomorphism from a group of order 10 to a group of order 6.

Q. 1. b) Express the ideal $4Z + 6Z$ as a principal ideal in the integral domain Z.

Sol^n:

To express the ideal $4Z + 6Z$ as a principal ideal in the integral domin Z (the set of all integers), we need to find a single integer "a" such that:

$$4Z + 6Z = aZ$$

We know that GCD of 4 and 6 is 2.

Now, we can use the GCD to find "a".

$$a = GCD(4,6) = 2$$

So, the ideal $4Z + 6Z$ can be expressed as a principal ideal in Z as:

$$4Z + 6Z = 2Z$$

In other words, the ideal is generated by the integer 2, and it consists of all multiples of 2 within the set of integers

Q 1 c) Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n)} \cdot \frac{x^{2n+1}}{2n+1}, \quad x>0$$

Sol^n:

let $u_n = \frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n)} \cdot \frac{x^{2n+1}}{2n+1}$ and

so $u_{n+1} = \frac{1.3.5.\ldots(2n-1)(2n+1)}{2.4.6.\ldots(2n)(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} \cdot \frac{1}{x^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n^2+10n+6}{4n^2+4n+1} \cdot \frac{1}{x^2} \quad - (1)$$

Now, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{4+(10/n)+(6/n^2)}{4+(4/n)+(1/n^2)} \cdot \frac{1}{x^2}$

$$= \frac{1}{x^2}$$

$\therefore$ By ratio test Σu_n converges if $\frac{1}{x^2} > 1$, ie. if $x^2 < 1$ and diverges if $\frac{1}{x^2} < 1$, i.e. if $x^2 > 1$. When $\frac{1}{x^2} = 1$, ie. $x^2 = 1$, the ratio test fails.

We shall now apply Rabbe's test.

from ①, for $x^2=1$,

$$\frac{u_n}{u_{n+1}} = \frac{4n^2+10n+6}{4n^2+4n+1}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)$$

$$= \lim_{n\to\infty} \frac{6n^2+5n}{4n^2+4n+1}$$

$$= \frac{3}{2} > 1$$

Hence, by Rabbe's test,

Σu_n converges for $x^2=1$

Thus the given series converges if $x^2 \leq 1$ and diverges if $x^2>1$.

Q. 1.d State the sufficient condition for a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ to be analytic in its domain. Hence, show that $f(z) = \log z$ is analytic in its domain and find $\frac{df}{dz}$. [10 Marks]

Sol^n:

Sufficient condition for $f(z)$ to be analytic

The single valued continuous function $f(z)$ is analytic in a domain D if the four partial derivatives u_x, v_x, u_y, v_y exist, are continuous and satisfy Cauchy-Riemann equation at each point of D.

Now, Given $f(z) = \log z$

let $f(z) = w = u(x,y) + iv(x,y)$

$$= \log(x+iy)$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

Then, $u(x,y) = \frac{1}{2}\log(x^2+y^2)$

$v(x,y) = \tan^{-1}\frac{y}{x}$

Now, $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$ and $\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ except at origin therefore u and v satisfy Cauchy - Riemann equation and all partial derivatives are continuous except at origin.

Hence $f(z) = \log z$ is analytic everywhere except at origin.

For $\frac{df}{dz}$

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

provided $z \neq 0$.

Q.1e) A person requires 24, 24 and 20 units of chemicals A, B, and C respectively for his garden. Product P contain 2, 4 and 1 units of chemical A, B and C respectively per jar and product Q contain 2, 1, and 5 units of chemicals A, B and C respectively per Jar. If a Jar of P cost ₹ 30 and a Jar of Q cost ₹ 50, then how many jars of each should be purchased in order to minimise cost and meet the requirements? [10M]

Sol^n let consider x & y are no. of P and Q product purchased respectively. cost of P and Q product are ₹ 30 and ₹ 50 respectively.

Hence objective function (z) is become

minimize $z_1 = 30x + 50y$.

Given condition -

chemical / Product	A	B	C	cost
P	2	4	1	30
Q	2	1	5	50
Requirement	24	24	20	

Minimise $z = 30x + 50y$

subject to $2x + 2y \geq 24$

$x + y \geq 12$

$4x + y \geq 24$

$x + 5y \geq 20$

and $x, y \geq 0$.

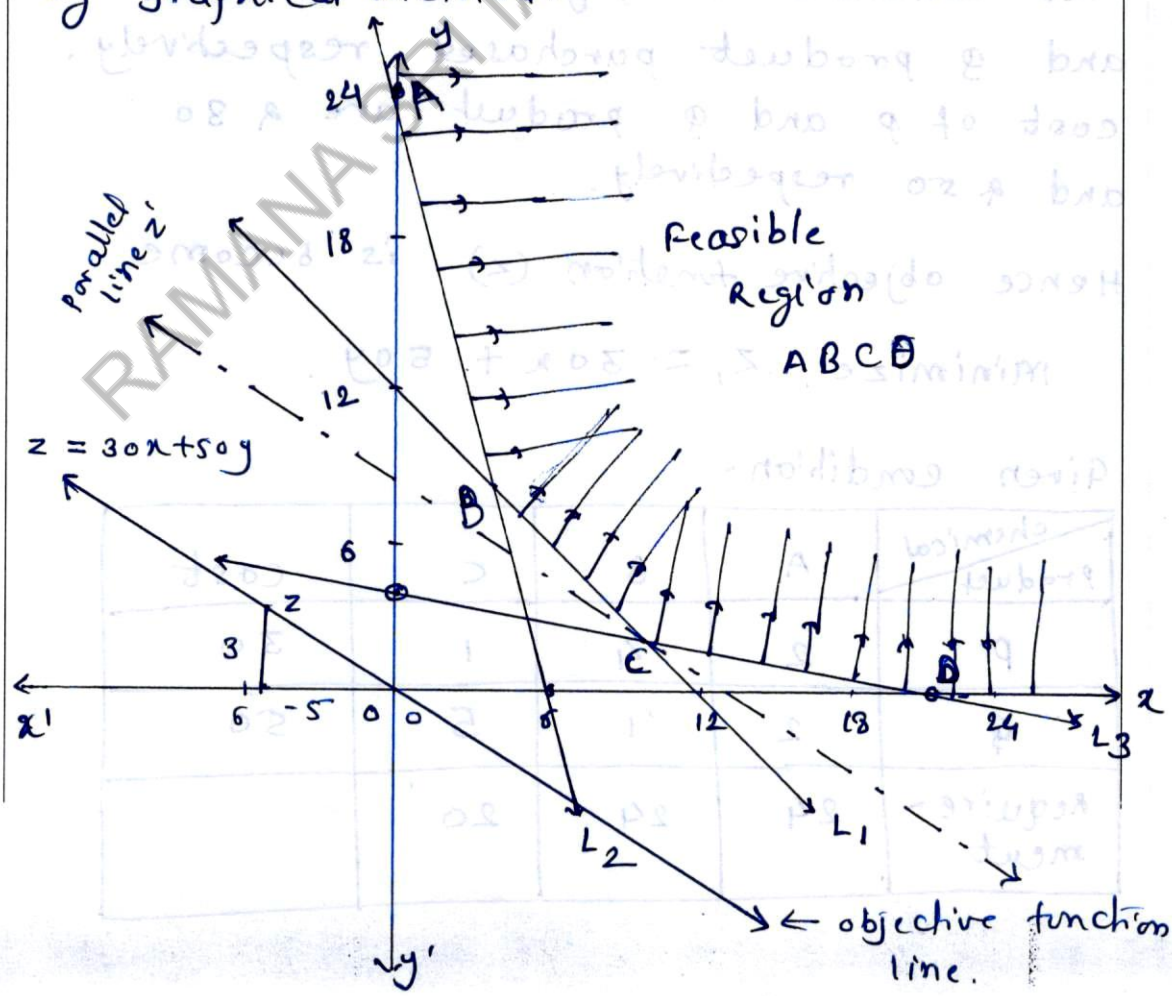
convert inequalities into equalities.

$x + y = 12$ —— line ①

$4x + y = 24$ —— line ②

$x + 5y = 20$ —— line ③

by graphical method.



In above graph the feasible region ABCD.

$z = 30x + 50y$.

$z = 0 \Rightarrow \frac{x}{y} = \frac{-5}{3}$

Draw the line through o and point z (-5, 3). this line is objective function line.

then continuously drawing line parallel to it till we reach nearest point of feasible region. because our objective function is minimization.

Zmin at point C. (10, 2) from graph.

$\min z = 30x + 50y$

$\min z = 30(10) + 50(2)$

$\min z = 400$

Hence we need to purchase 10 unit and 2 unit of product p & product q in order to minimise cost.

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Candidates must not write on this margin

Q.2 a Prove that a non-commutative group of order $2p$, where p is an odd prime, must have a subgroup of order p. [15M].

sol^n → Given $(G, \cdot)$ be commutative group with order $2p$.

$$\Rightarrow o(G) = 2p$$

where p is odd prime

we know the prime number

$p = 2, 3, 5 \ldots\ldots$ p is odd prime

$\Rightarrow p > 2$

$\boxed{\text{case I}}$

If G is cyclic group of order $2p$

we know that every cyclic group is abelian group. Hence G is abelian group.

But Given G is non-commutative group i.e non-abelian group. Hence our assumption wrong. G cannot be cyclic group.

$\boxed{\text{case II}}$ - G is non-cyclic group of order $2p$.

$\boxed{\text{sub case (i)}}$ — If there is an element a of order p then (a) is a subgroup of order p.

$o(a) = p$

$H = \{a^1, a^2, \ldots\ldots a^p = e\}$ is order p subgroup.

Hence ~~proved~~ prove require result.

$\boxed{\text{sub case (ii)}}$ -

If there is no element of order p.

Then by Lagranges theorem possible order of subgroup are $1, 2, p, 2p$.

we assume no subgroup of order p.

a) $o(a) = 1 \Rightarrow a^1 = e$

$H = \{e\}$ this is improper subgroup.

b) $o(a) = 2 \quad \forall\ a \in G$

$a^2 = e$, $b^2 = e$, let $a, b \in G$

$\Rightarrow ab \in G$ (as closure property let

$ab = (ab)^{-1} = b^{-1}a^{-1} = ba$

$\because ab = ba \Rightarrow$ group is abelian group.

which contradict, as given G is non-commutative group.

c) $o(G) = 2p$

$o(a) = 2p$ & $o(G) = 2p$

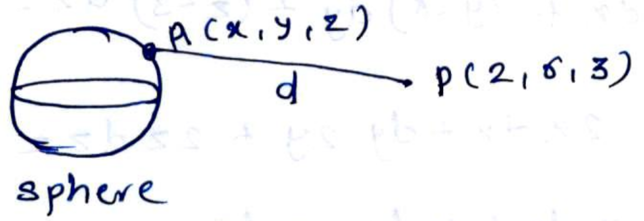
$G = \langle a \rangle$ l(a)

hence G is cyclic group. which contradict our assumption G is non-cyclic group.

from above case ① & case ② we conclude that if G is non-commutative group of order 2p, p is odd prime then there must be subgroup of order p.

Q.2 b) Using the method of Lagrange's Multipliers, find the minimum and maximum distances of the points $p(2,6,3)$ from the sphere $x^2+y^2+z^2=4$.

Sol^n →



let point $A(x,y,z)$ be any point on sphere. $p(2,6,3)$ is given point.

$$pA^2 = (x-2)^2+(y-6)^2+(z-3)^2 \quad \text{——} (1)$$

...... [by using distance formula

distance between two point (x_1,y_1,z_1) and $(x_2\, y_2\, z_2)$ is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

we have to find the maximum and minimum value of $f(x,y,z)$ subject to the given condition

$$\phi(x,y,z) \equiv x^2+y^2+z^2-4=0 \quad \text{——} (2)$$

Now let, maximum & minimum,

$$U = pA^2 = (x-2)^2+(y-6)^2+(z-3)^2$$

subject to $\phi \equiv x^2+y^2+z^2-4=0$.

by Lagrange's multipliers method.

$dU = 0 \Rightarrow 2(x-2)\,dx + 2(y-6)\,dy + 2(z-3)\,dz = 0$

$(x-2)\,dx + (y-6)\,dy + (z-3)\,dz = 0$ — ③

$d\phi = 0 \Rightarrow 2x\,dx + dy\,2y + 2z\,dz = 0$

$x\,dx + y\,dy + z\,dz = 0$ — ④

eq^n ③ $\times 1 + \lambda \times eq^n$ ④

$dU + \lambda\, d\phi = 0$

$(x-2)\,dx + (y-6)\,dy + (z-3)\,dz + \lambda(x\,dx + y\,dy + z\,dz) = 0$ — ⑤

$(x-2) + \lambda x = 0$ — ⑥

$(y-6) + \lambda y = 0$ — ⑦

$(z-3) + \lambda z = 0$ — ⑧

from eq^n ⑥ ⑦ & ⑧.

$$\frac{x-2}{x} = \frac{y-6}{y} = \frac{z-3}{z} = \lambda$$

$$= \frac{\sqrt{(x-2)^2 + (y-6)^2 + (z-3)^2}}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{x-2}{x} = \frac{y-6}{y} = \frac{z-3}{z} = \pm\frac{\sqrt{U}}{2} = \lambda$$

$$\lambda = \pm\frac{\sqrt{U}}{2}.$$

substitute λ value in (6) (7) & (8).

$$x = \frac{2}{1+\lambda} \qquad y = \frac{6}{1+\lambda} \qquad z = \frac{3}{1+\lambda}$$

$$x = \frac{2}{1 \pm \frac{\sqrt{u}}{2}} \qquad y = \frac{6}{1 \pm \frac{\sqrt{u}}{2}} \qquad z = \frac{3}{1 + \frac{\sqrt{u}}{2}}$$

substitute value of x, y, z in eqn (2)

$$x^2 + y^2 + z^2 = 4$$

$$\frac{4}{\left(1 \pm \frac{\sqrt{u}}{2}\right)^2} + \frac{36}{\left(1 \pm \frac{\sqrt{u}}{2}\right)^2} + \frac{9}{\left(1 + \frac{\sqrt{u}}{2}\right)^2} = 4$$

$$4 \times \left[\frac{2 \pm \sqrt{u}}{2}\right]^2 = 4 + 36 + 9$$

$$(2 \pm \sqrt{u})^2 = 49$$

$$2 \pm \sqrt{u} = \pm 7$$

$$\pm \sqrt{u} = 2 \pm 7$$

$$\sqrt{u} = 9 \quad \text{or} \quad \sqrt{u} = -5$$

squaring $u = 81$ & $u = 25$.

hence maximum value & minimum value.

$$\boxed{U_{max} = 81} \qquad \boxed{U_{min} = 25}.$$

Maximimum & minimum distance of point $P(2, 6, 3)$ from sphere $x^2 + y^2 + z^2 = 4$ is 81 unit & 25 unit.

Q.2.c Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$ using contour Integration [20 Marks]

Soln: Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

$$= \text{real part of} \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta} d\theta$$

$$= \text{real part of} \int_C \frac{z^2}{5+2\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= \text{real part of } \frac{1}{i}\int_C \frac{z^2}{2z^2+5z+2} dz$$

$$= \text{real part of } \frac{1}{2i}\int_C \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)} dz$$

$$= \text{real part of} \int_C f(z)\, dz \quad (\text{say})$$

where c is the unit circle.

Now, $z = \frac{-1}{2}, -2$ are the simple poles of $f(z)$. Out of these only $z = \frac{-1}{2}$ lies inside c.

Residue at $z = \frac{-1}{2}$ is

$$\lim_{z \to \frac{-1}{2}} \left(z+\frac{1}{2}\right) f(z)$$

$$= \lim_{z \to \frac{-1}{2}} \left(z+\frac{1}{2}\right) \frac{z^2}{2i\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \lim_{z \to \frac{-1}{2}} \frac{z^2}{2i(z+2)} = \frac{1}{12i}$$

$$\therefore \int_C f(z)\,dz = 2\pi i \cdot \frac{1}{12i}$$

$$= \frac{\pi}{6}$$

Hence $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

$$= \text{real part of } \int_C f(z)\,dz = \frac{\pi}{6}$$

Q. 3 a) Prove that x^2+1 is an irreducible polynomial in $Z_3[x]$. Further show that the quotient ring $\frac{Z_3[x]}{\langle x^2+1\rangle}$ is a field of 9 elements.
[15 M]

Sol^n: To prove that x^2+1 is an irreducible polynomial in $Z_3[x]$, we need to show that it cannot be factored into non-constant polynomials over the field Z_3. We can do this by considering all possible factorizations.

First, let's check if x^2+1 has any linear factors in $Z_3[x]$. We can do this by substituting all elements of Z_3 (which are 0, 1 and 2) into x^2+1 and checking if any of them equals 0:

for $x=0$, we have $0^2+1=1$
for $x=1$, we have $1^2+1=2$
for $x=2$, we have $2^2+1=5$, but since we're working in Z_3, $5\equiv 2 \pmod 3$.

None of the elements in Z_3 makes x^2+1 equal to 0. Therefore, x^2+1 has no linear factors in $Z_3[x]$.

Now, let's consider the possibility of x^2+1 having quadratic factors. We'll assume it has a quadratic factor $ax+b$, where a and b are in Z_3 & a is non-zero. This means that x^2+1 should be divisible by $(ax+b)^2$

Expanding $(ax+b)^2$, we get $a^2x^2 + 2abx + b^2$.
To be a factor of x^2+1, this should satisfy:

$$a^2x^2 + 2abx + b^2 \equiv x^2+1 \pmod 3$$

Comparing coefficients, we get two equations:

$$a^2 \equiv 1 \pmod 3 \quad - (1)$$

$$2ab \equiv 0 \pmod 3 \quad - (2)$$

from (1), a can be either 1 or 2
(since $(-1)^2 \equiv 1 \pmod 3$)

from (2), If a is 1, then $2b \equiv 0 \pmod 3$, which implies b must be 0.
If a is 2, then $4b \equiv 0 \pmod 3$, which again implies b must be 0.

Therefore, the only possibility for a and b is $a=1$ and $b=0$, which corresponds to the linear factor x. But we have already shown that x^2+1 has no linear factors.

Since x^2+1 has no linear or quadratic factors in $Z_3[x]$, it is irreducible in $Z_3[x]$.

Now, let's consider the quotient ring $\frac{Z_3[x]}{(x^2+1)}$
This ring is essentially the set of all polynomials in $Z_3[x]$ modulo the ideal generated by x^2+1.

In this quotient ring, any polynomial of degree 2 or higher can be reduced to a polynomial of degree at most 1 using the relation $x^2 \equiv -1 \pmod 3$, which comes from the ideal (x^2+1).

The elements of this quotient ring can be represented as $a+bx$, where a and b are in Z_3.

There are 3 choices for a $(0,1,2)$ and 3 choices for b $(0,1,2)$ leading to a total of 9 elements in the quotient ring.

So, the quotient ring $\frac{Z_3[x]}{(x^2+1)}$ is indeed a field of 9 elements.

Q. 3.b Prove that $u(x,y) = e^x(x\cos y - y\sin y)$ is harmonic. Find its conjugate harmonic function $v(x,y)$ and express the corresponding analytic function $f(z)$ in terms of z. [15 Marks]

Soln:

Given $u = e^x(x\cos y - y\sin y)$

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y + \cos y)$$

$$\frac{\partial u}{\partial y} = -e^x(x\sin y + y\cos y + \sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x(2\cos y + x\cos y - y\sin y)$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x(2\cos y + x\cos y - y\sin y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (2\cos y + x\cos y - y\sin y)\cdot(e^x - e^x)$$

$$= 0$$

$\therefore$ u is harmonic.

Let v be the harmonic conjugate of u.

$\Rightarrow f(z) = u + iv$ is analytic

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$ { By Cauchy-Riemann eq^n $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$= e^x(\cos y + x\cos y - y\sin y) + ie^x(x\sin y + \sin y + y\cos y)$$

Applying Milne-Thomson's method

putting $x = z$ & $y = 0$ we get

$$f'(z) = e^z(1+z) + ie^z(0)$$

$$= e^z(1+z)$$

Integrating w.r.t 'z'

$$f(z) = e^z(1+z) - (1)e^z$$

$$= ze^z$$

$\therefore$ Analytic function in terms of z

$$\boxed{f(z) = ze^z}$$

Now, $z = x+iy$

$$\therefore f(z) = (x+iy)e^{x+iy}$$

$$= e^x(x+iy)(\cos y + i\sin y)$$

$[\because e^{iy} = \cos y + i\sin y]$

$$= e^x[x\cos y - y\sin y] + ie^x[x\sin y + y\cos x]$$

$\therefore$ Conjugate $\boxed{V = e^x(x\sin y + y\cos x)}$

Q.3 c) Solve the following linear programming Problem by Big M method :

Minimise $z = 2x_1 + 3x_2$

subject to $x_1 + x_2 \geq 9$

$x_1 + 2x_2 \geq 15$

$2x_1 - 3x_2 \leq 9$

$x_1, x_2 \geq 0$

is the optimal solution unique ? Justify your answer.

Sol^n $\rightarrow$

$z^* = -z$

then objective function.

maximise $z^* = -2x_1 - 3x_2$

convert inequalities into equalities.

$x_1 + x_2 - S_1 + A_1 = 9$

$x_1 + 2x_2 - S_2 + A_2 = 15$

$2x_1 - 3x_2 + S_3 = 9$

$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0.$

hence objective function given by

$z^* = -2x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$

i) Basic solution

non-basic variable $\Rightarrow x_1 = x_2 = S_1 = S_2 = 0$

& Basic variable $\Rightarrow A_1 = 9 \quad A_2 = 15$

$S_3 = 9$

hence B.S is $(0, 0, 0, 0, 9, 9, 15)$.

B	C_B	c_j	-2	-3	0	0	0	$-M$	$-M$		
		x_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2		Ratio
A_1	$-M$	9	1	1	-1	0	0	1	0		9
A_2	$-M$	15	1	2	0	-1	0	0	1	←	7.5
S_3	0	9	2	-3	0	0	1	0	0		–
		Δ_j	$2M-2$	$3M-3$ ↑	$-M$	0	0	0	0		
A_1	$-M$	$3/2$	$1/2$	0	-1	$-1/2$	0	1			
x_2	-3	$15/2$	$1/2$	1	0	$-1/2$	0	0			
S_3	0	$63/2$	$7/2$	0	0	$-3/2$	1	0			
		Δ_j	$\frac{3M}{2}-2$	-3	M	$\frac{M}{2}-\frac{3}{2}$	0	M			
x_1	-2	3	1	0	-2	1	0				
x_2	-3	6	0	1	$1/2$	-1	0				
S_3	0	21	0	0	7	-5	1				
		Δ_j	0	0	$-\frac{5}{2}$	-1	0				

⇒ since all Δ_j's are zero or negative, so the solution is optimal.

optimal solution is

$x_1 = 3$, $x_2 = 6$, $s_1 = 0$, $s_2 = 0$, $s_3 = 0$

$A_1 = 0$ $A_2 = 0$.

Maximise $z^* = -2x_1 - 3x_2$

$z^* = -2\times 3 - 3\times 6$

$z^* = -24$.

hence required solution

Minimise $z = -z^*$. Minimum value $z = 24$.

optimal solution unique or not —

since Δ_j coloumn of non basic variable are zero.

non basic variable x_1, x_2, s_1, s_2 in Δ_j column are $0, 0, -5/2, -1$.

Hence we can conclude that alternate solution is available.

Hence above LPP problem's optimal solution is not unique.

Q. 4 a) Prove that the oscillation of a real-valued bounded function f defined on $[a,b]$ is the supremum of the set $\{|f(x_1)-f(x_2)| : x_1, x_2 \in [a,b]\}$ [15 M]

Sol^n: To prove that the oscillation of a real-valued bounded function f defined on the $[a,b]$ is the supremum of the set $\{|f(x_1)-f(x_2)| : x_1, x_2 \in [a,b]\}$, we'll have to first show that the supremum exists and is indeed the oscillation.

Let $A = \{|f(x_1)-f(x_2)| : x_1, x_2 \in [a,b]\}$ and let M be the supremum of A. We want to prove that M is the oscillation of

Existence of supremum :

Since f is bounded on $[a,b]$, there exists a no. $K>0$, such that $-K \leq f(x) \leq K \quad \forall\, x \in [a,b]$.

Therefore, for any $x_1, x_2 \in [a,b]$, we have :

$$-K \leq f(x_1) - f(x_2) \leq K$$

$$\Rightarrow \quad |f(x_1)-f(x_2)| \leq 2K \quad \forall\, x_1, x_2 \in [a,b].$$

Thus, A is bounded from above, and by the completeness property of real numbers, A has a supremum.

Showing that M is the oscillation :

To prove that M is the oscillation, we need to show two things :

i) M is an upper bound of A.

ii) For any $\epsilon > 0$, $\exists\ x_1, x_2 \in [a,b]$ such that

$$|f(x_1) - f(x_2)| > M - \epsilon$$

Let's prove these two things :

i) Since M is the supremum of A, by definition, M is an upper bound of A.

ii) Now, for any $\epsilon > 0$, consider the number $M - \epsilon/2$. By the definition of supremum, $\exists\ x_1, x_2 \in [a,b]$ such that

$$M - \epsilon/2 < |f(x_1) - f(x_2)| \leq M$$

Now, we choose x_1 and x_2 as the points for which the supremum is achieved :

$$|f(x_1) - f(x_2)| = M$$

Since $M - \epsilon/2 < M$, it follows that

$$|f(x_1) - f(x_2)| > M - \epsilon/2 \geq M - \epsilon$$

$\therefore$, for any $\epsilon > 0$, we can find $x_1, x_2 \in [a,b]$ such that $|f(x_1) - f(x_2)| > M - \epsilon$

Combining points i) and ii), we have shown that M is the oscillation of the function f on the interval $[a,b]$. This completes the proof.

Q. 4. b Classify the singular point $z=0$ of the function $f(z) = \frac{e^z}{z - \sin z}$ and obtain the principal part of its Laurent series expansion.

Soln: Given $f(z) = \frac{e^z}{z-\sin z}$

To find singularity equate $z - \sin z = 0$

$$\Rightarrow z - z + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} \cdots = 0$$

$$z^3\left(\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots\right) = 0 \quad -(1)$$

$$\Rightarrow z^3 = 0$$

$\therefore$ $z=0$ is singular point

& $z=0$ is pole of $f(z)$ of order 3.

Now,

$$f(z) = \frac{e^z}{z-\sin z}$$

$$= \frac{e^z}{\frac{z^3}{3!}\left(1 - \frac{z^2}{20} + \frac{z^4}{840} - \cdots\right)} \quad \left\{\because \text{from } (1)\right.$$

$$= \frac{3!}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)(1-k)^{-1}$$

where $k = \frac{z^2}{20} - \frac{z^4}{840} + \cdots$

$$= \frac{6}{z^3}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}\cdots\right)\left(1+k+k^2\cdots\right)$$

$$= \left(\frac{6}{z^3}+\frac{6}{z^2}+\frac{3}{z}\cdots\right)\left(1+\frac{z^2}{20}+\frac{z^4}{840}\cdots\right)$$

$$= \frac{6}{z^3}+\frac{6}{z^2}+\frac{33}{10z}+\frac{6}{20}\cdots$$

Hence the principal part of its Laurent series expansion is

$$\boxed{\frac{6}{z^3}+\frac{6}{z^2}+\frac{33}{10z}}$$

Q.4c A department head has 5 subordinates and 5 jobs to be performed. The time (in hours) that each subordinate will take to perform each job is given in the matrix below.

Jobs

Subordinate	A	B	C	D	E
I	4	9	4	12	4
II	15	11	20	5	8
III	17	7	15	12	18
IV	9	13	11	9	14
V	6	11	12	9	14

How should the jobs be assigned, one to each subordinate, so as to minimize the total time? Also, obtain the total minimum time to perform all the jobs if the subordinate IV cannot be assigned job C. [20M]

$sol^n \rightarrow$ Given subordinate IV cannot be assigned job C. Hence we put in that cell put ∞ as allocation value. hence time matrix given by as below.

4	9	4	12	4
15	11	20	5	8
17	7	15	12	18
9	13	∞	9	14
6	11	12	9	14

Now proceeding with Hungarian method.
finding row min and subtract row min from all other element.
row min are - 4, 5, 7, 9, 6.

0	5	0	8	0
10	6	15	0	3
10	0	8	5	11
0	4	∞	0	5
0	5	6	3	8

coloumn min - 0, 0, 0, 0, 0. subtract from coloumn element. we will get same matrix.

Now, assignment start.

$\cancel{0}$	5	$\boxed{0}$	8	$\cancel{0}$
10	6	15	$\boxed{0}$	3
10	$\boxed{0}$	8	5	11
$\boxed{0}$	4	∞	$\cancel{0}$	5
$\cancel{0}$	5	6	3	8

As no. of assignment are four only. no. of subordinate are five. hence not optimal.

$\cancel{0}$	5	$\boxed{0}$	8	$\cancel{0}$
10	6	15	$\boxed{0}$	3
10	$\boxed{0}$	8	5	11
$\boxed{0}$	4	∞	$\cancel{0}$	5
$\cancel{0}$	5	6	3	8

by Hungarian method, new matrix will be

3	5	$\boxed{0}$	11	$\cancel{0}$
10	3	13	10	$\boxed{0}$
13	$\boxed{0}$	8	8	11
$\cancel{0}$	1	∞	$\boxed{0}$	2
$\boxed{0}$	5	6	3	8

now no. of assignment are five. hence we reached to optimal solution. & solution is.

Assignment are.

subordinate	I	II	III	IV	V
Job	C	E	B	D	A
time	4	8	7	9	6

$\therefore$ Hence total time = 34 Hours.

Q. 5 a) By eliminating the arbitrary functions f and g from $z = f(x^2-y) + g(x^2+y)$, form partial differential equation. (10 M)

Sol^n: Given $z = f(x^2-y) + g(x^2+y)$ — ①

differentiating ① w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = 2x f'(x^2-y) + 2xg'(x^2+y)$$

$$\therefore \frac{\partial z}{\partial x} = 2x\left[f'(x^2-y) + g'(x^2+y)\right] \quad -② $$

$$\frac{\partial z}{\partial y} = -f'(x^2-y) + g'(x^2+y) \quad -③$$

differentiating ② and ③ w.r.t. x and y respectively, we get

$$\frac{\partial^2 z}{\partial x^2} = 2\left[f'(x^2-y) + g'(x^2+y)\right] + 4x^2\left[f''(x^2-y) + g''(x^2+y)\right] \quad -④$$

and $$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y) \quad -⑤$$

Again ② $\Rightarrow$ $$f'(x^2-y) + g'(x^2+y) = \frac{1}{2x}\left(\frac{\partial z}{\partial x}\right) \quad -⑥$$

$\therefore$ Substituting values of $f''(x^2-y) + g''(x^2+y)$ and $f'(x^2-y) + g'(x^2+y)$ from ⑤ & ⑥ in ④,

we have, $$\frac{\partial^2 z}{\partial x^2} = 2x\frac{1}{2x}\left(\frac{\partial z}{\partial x}\right) + 4x^2\frac{\partial^2 z}{\partial y^2}$$ or

$$\boxed{x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2}}$$

↳ which is required PDE.

5 (b) Given $\frac{dy}{dx} = \frac{y^2-x}{y^2+x}$ with initial condition $y=1$ at $x=0$. find the value of y for $x=0.4$ by Euler's Method, correct to 4 decimal places, taking step length $h=0.1$ [10 M]

sol^n

$$f(x,y) = \frac{dy}{dx} = \frac{y^2-x}{y^2+x}$$

$y(0)=1$ $y(0.4)=?$ $h=0.1$

we know that, Euler method

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

$i=0$ $x_0=0$ $h=0.1$ $y_0=1$

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + 0.1\left(\frac{1^2-0}{1^2+1}\right)$$
$$= 1 + 0.1$$
$$= 1.1$$

$y_1 = 1.1$

similar way proceeding we get following table.

x_i	y_i	$f(x_i, y_i)$	$y_{i+1} = y_i + h f(x_i, y_i)$
0	1	1	1.1
0.1	1.1	0.847328	1.18473
0.2	1.18473	0.75056	1.25978
0.3	1.25978	0.68204	1.327989

hence required value

$$y(0.4) = 1.327989$$

$$\boxed{y(0.4) = 1.3280}$$

Q.5 c Evaluate, using the binary arithmetic, the following numbers in their given system:

i) $(634.235)_8 - (132.223)_8$

ii) $(7AB.432)_{16} - (5CA.D61)_{16}$

sol^n $\Rightarrow (634.235)_8 = (110\ 011\ 100.010\ 011\ 001)_2$

$(132-223)_8 = (001\ 011\ 010.010\ 010\ 011)_2$

using 1's complementary method of subtraction

$$\begin{array}{lr} x & -\ 110\ 011\ 100.010\ 011\ 001 \\ + \ 1\text{'s of } y & -\ 110\ 100\ 101.101\ 101\ 100 \\ \hline & \boxed{*}\ 101\ 000\ 010.000\ 000\ 101 \\ & +\qquad\qquad\qquad\qquad 1 \\ \hline & (101\ 000\ 010.000\ 000\ 110)_2 \end{array}$$

$(101\ 000\ 010.000\ 000\ 110)_2 = (502.006)_8$

hence require solution is

$$\boxed{(634.235)_8 - (132.223)_8 = (502.006)_8}$$

ii) $(7AB.432)_{16} = 0111\ 1010\ 1011.0100\ 0011\ 0010$

$(5CA.D61)_{16} = 0101\ 1100\ 1010.1101\ 0110\ 0001$

using 1's compliment method

x – 0111 1010 1011 . 0100 0011 0010

\+ 1's of y – 1010 0011 0101 . 0010 1001 1110

$\boxed{1}$ 0001 1110 0000 . 0110 1101 0000

\+ 1

$(0001\ 1110\ 0000 . 0110\ 1101\ 0001)_2$

in hexadecimal

1 E 0 . 6 D 1

i.e $(1E0.6D1)_{16}$

hence require solution is

$(7AB.432)_{16} - (5CA.D61)_{16} = (1E0.6D1)_{16}$

is required solution.

Note – Verification by decimal

i) 412.3066
 – 90.2871
 322.019 — $(502.006)_8$

ii) 1963.2622
 1482.8361
 480.4260 – $(1E0.6D1)_{16}$

Q. 5. d) A planet of mass m is revolving around the sun of mass M. The kinetic energy T and potential energy V of the planet are given by $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$ and $V = GMm\left(\frac{1}{2a} - \frac{1}{r}\right)$, where (r, θ) are the polar co-ordinates of the planet at time t, G is the gravitational constant and $2a$ is the major axis of the ellipse (the path of the planet). Find the Hamiltonian and the Hamilton equations of the planet's motion. (10 M)

Sol^n: i) We have,

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) \quad \text{- given}$$

and $$V = GMm\left(\frac{1}{2a} - \frac{1}{r}\right) \quad \text{- given}$$

we know that

$$L = T - V$$

$$\therefore \quad L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - GMm\left(\frac{1}{2a} - \frac{1}{r}\right)$$

$$\therefore \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\Rightarrow \quad \dot{r} = \frac{p_r}{m} \quad \text{and} \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Thus $$H = \Sigma p_i \dot{q}_i - L$$

$$= p_r \dot{q}_r + p_\theta \dot{q}_\theta - L$$

$$= p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$+ GMm\left(\frac{1}{2a} - \frac{1}{r}\right)$$

$$= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) - \frac{1}{2}m\left[\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta^2}{m^2r^4}\right)\right]$$

$$+ GMm\left(\frac{1}{2a} - \frac{1}{r}\right)$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + GMm\left(\frac{1}{2a} - \frac{1}{r}\right)$$

= total energy of the system.

$$\therefore \boxed{H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + GMm\left(\frac{1}{2a} - \frac{1}{r}\right)}$$

ii) Hamilton's equations are

$$\dot{p}_i = -\left(\frac{\partial H}{\partial q_i}\right), \quad \dot{q}_i = \left(\frac{\partial H}{\partial p_i}\right)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{p}_i = -\left(\frac{\partial H}{\partial r}\right) = \frac{p_\theta^2}{mr^3} - GMm\left(\frac{1}{2a} - \frac{1}{r}\right)$$

$$\dot{p}_\theta = -\left(\frac{\partial H}{\partial \theta}\right) = 0$$

which is required solution.

Q. 5 e) In a fluid motion, there is a source of strength ~~of~~ 2m placed at $z=2$ and two sinks of strength m are placed at $z=2+i$ and $z=2-i$. find the streamlines. (10 M)

Sol^n: Given that one source strength '2m' is placed at $z=2$ and two sinks of strength 'm' are placed at $z=2+i$ and $z=2-i$.

$\therefore$ Complex potential due to above

$$w = 2m\log(z-2) - m\log\{z-(2+i)\} - m\log\{z-(2-i)\}$$

$$w = m\{\log(z-2)^2 - \log(z-2-i) - \log(z-2+i)\}$$

$$\phi + i\psi = m\{\log[(x-2)+iy]^2 - \log[(x-2)+i(y-1)] - \log[(x-2)+i(y+1)]\}$$

$$\psi = m\tan^{-1}\left[\frac{2y(x-2)}{(x-2)^2-y^2}\right] - \tan^{-1}\left[\frac{y-1}{x-2}\right] - \tan^{-1}\left[\frac{y+1}{x-2}\right]$$

$\therefore$ for streamlines $\psi = c$, where c is constant

$$\Rightarrow \quad c = m\tan^{-1}\left[\frac{2y(x-2)}{(x-2)^2-y^2}\right] - \tan^{-1}\left[\frac{\frac{y-1}{x-2}+\frac{y+1}{x-2}}{1-\left(\frac{y-1}{x-2}\times\frac{y+1}{x-2}\right)}\right]$$

$$c = m\tan^{-1}\left[\frac{2y(x-2)}{(x-2)^2-y^2}\right] - \tan^{-1}\left[\frac{2y(x-2)}{(x-2)^2-(y^2-1)}\right]$$

$$\frac{c}{m} = \tan^{-1}\left[\frac{2y(x-2)}{x^2-y^2-4x+4}\right] - \tan^{-1}\left[\frac{2y(x-2)}{x^2-y^2-4x+5}\right]$$

$$c = \tan^{-1}\left[\frac{\frac{2y(x-2)}{x^2-y^2-4x+4} - \frac{2y(x-2)}{x^2-y^2-4x+5}}{1+\left(\frac{2y(x-2)}{x^2-y^2-4x+4}\right)\left(\frac{2y(x-2)}{x^2-y^2-4x+5}\right)}\right]$$

$\left(\because \frac{c}{m} \text{ is constant}\right)$

$$\tan c = \left[\frac{\{(x^2-y^2-4x+5)-(x^2-y^2-4x+4)\}\, 2y(x-2)}{(x^2-y^2-4x+4)(x^2-y^2-4x+5)+4y^2(x-2)^2}\right]$$

$$c = \frac{(1)\, 2y(x-2)}{(x^2-y^2-4x+4)(x^2-y^2-4x+5)+4y^2(x-2)^2}$$

$(\because \tan c \text{ is constant})$

$$\Rightarrow c = \frac{y(x-2)}{(x^2-y^2-4x+4)(x^2-y^2-4x+5)+4y^2(x-2)^2}$$

or $(x^2-y^2-4x+4)(x^2-y^2-4x+5)+4y^2(x-2)^2 = cy(x-2)$

or $\boxed{x^4+y^4+2x^2y^2-8xy^2+25x^2+7y^2-36x+20 = cy(x-2)}$

Q. 6. a) Find the surface passing through the two lines $z = x = 0$ and $z - 1 = x - y = 0$, and satisfying the partial differential equation $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0$ [15 M]

Sol^n: Given: $z = x = 0$ and $z - 1 = x - y = 0$

and pde :- $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0$

given equation can be written as

$$(D^2 - 4DD' + 4D'^2)z = 0$$

$$\Rightarrow \quad (D - 2D')^2 z = 0$$

$\therefore$ Its solution is

$$z = \phi_1(y+2x) + x\,\phi_2(y+2x) \quad -① $$

where ϕ_1, ϕ_2 being arbitrary function

Since eq^n ① passes through $z = x = 0$

then $0 = \phi_1(y)$ which gives

$$\phi_1(y+2x) = 0$$

$\therefore$ Eq^n ① becomes $z = x\,\phi_2(y+2x) \quad - ②$

Since ② passes through $z - 1 = x - y = 0$

i.e. $z = 1$ & $y = x$, we get

$1 = x\,\phi_2(3x)$ or $\phi_2(3x) = \frac{3}{(3x)} \Rightarrow \phi_2(x) = \frac{3}{x}$

$\therefore \phi_2(y+2x) = 3/(y+2x)$

$\therefore$ from ②, we have $\boxed{3x = z(y+2x)}$

which is required surface

Q.6 b Solve the system of linear equations

$$7x_1 - x_2 + 2x_3 = 11$$
$$2x_1 + 8x_2 - x_3 = 9$$
$$x_1 - 2x_2 + 9x_3 = 7$$

correct upto 4 significant figures by the Gauss-Seidel iterative method. Take initially guessed solution as $x_1 = x_2 = x_3 = 0$.

[15M].

Sol^n → Above Given eq^n are arrange in complete pivoting. Hence proceeding.

$$x_1^{k+1} = \frac{11 + x_2^k - 2x_3^k}{7}$$

$$x_2^{k+1} = \frac{9 - 2x_1^{k+1} + x_3}{8}$$

$$x_3^{k+1} = \frac{7 - x_1^{k+1} + 2x_2^{k+1}}{9}$$

Initial value $x_1^0 = x_2^0 = x_3^0 = 0$

$k=1$ $\quad x_1^1 = \dfrac{11 + 0 - 2(0)}{7} = 1.571428$

$$x_2^1 = \frac{9 - 2(1.571428) + 0}{8} = 0.732143$$

$$x_3^1 = \frac{7 - 1.571428 + 2(0.732143)}{9} = 0.765873.$$

Similar way proceeding, $k=2$

$x_1^2 = 1.457199$

$x_2^2 = 0.856494$

$x_3^2 = 0.806185$

$k = 3$, $x_1^3 = 1.463437$

$x_2^3 = 0.8599138$

$x_3^3 = 0.8062656$

$k = 4$ $x_1^4 = 1.4639118$

$x_2^4 = 0.866511$

$x_3^4 = 0.80767$

$k = 5$ $x_1^5 = 1.464453$

$x_2^5 = 0.8598455$

$x_3^5 = 0.806137$.

hence require solution correct upto 4 decimal point is

$x_1 = 1.46445$

$x_2 = 0.85985$

$x_3 = 0.80614$.

Q. 6 c) A mechanical system with 2 degrees of freedom has the Lagrangian

$$L = \frac{1}{2}m(\dot x^2 + \dot y^2) - \frac{1}{2}m(\omega_1^2 x^2 + \omega_2^2 y^2) + kxy$$

where m, ω_1, ω_2, k are constants. find the parameter θ so that under the transformation

$$x = q_1\cos\theta - q_2\sin\theta, \quad y = q_1\sin\theta + q_2\cos\theta$$

the Lagrangian in terms of q_1, q_2 will not contain the product term $q_1 q_2$. find the Lagrange's equations w.r.t. q_1 and q_2 independent of parameter θ. [20 M]

Sol^n: Given that,

$$x = q_1\cos\theta - q_2\sin\theta$$

$$y = q_1\sin\theta + q_2\cos\theta$$

differentiating x and y;

$$\dot x = \dot q_1\cos\theta - \dot q_2\sin\theta - q_1\sin\theta\dot\theta - q_2\cos\theta\dot\theta$$

$$\dot y = \dot q_1\sin\theta + \dot q_2\cos\theta + q_1\cos\theta\dot\theta - q_2\sin\theta\dot\theta$$

$$\dot x^2 = \dot q_1^2\cos^2\theta + \dot q_2^2\sin^2\theta + q_1^2\sin^2\theta\dot\theta^2 + q_2^2\cos^2\theta\dot\theta^2$$
$$- 2\dot q_1\dot q_2\sin\theta\cos\theta - 2q_1\dot q_1\sin\theta\cos\theta\dot\theta - 2\dot q_1 q_2\cos^2\theta\dot\theta$$
$$+ 2q_1\dot q_2\sin^2\theta\dot\theta + 2q_2\dot q_2\sin\theta\cos\theta\dot\theta + 2q_1q_2\sin\theta\cos\theta\dot\theta^2$$

$$\dot y^2 = \dot q_1^2\sin^2\theta + \dot q_2^2\cos^2\theta + q_1^2\cos^2\theta\dot\theta^2 + q_2^2\sin^2\theta\dot\theta^2$$
$$+ 2\dot q_1\dot q_2\sin\theta\cos\theta + 2\dot q_1 q_1\sin\theta\cos\theta\dot\theta - 2\dot q_1 q_2\sin^2\theta\cdot\dot\theta$$
$$+ 2q_1\dot q_2\cos^2\theta\dot\theta - 2q_2\dot q_2\sin\theta\cos\theta\dot\theta - 2q_1q_2\sin\theta\cos\theta\dot\theta^2$$

$$\dot{x}^2+\dot{y}^2 = \dot{q}_1^2+\dot{q}_2^2+(q_1^2+q_2^2)\dot{\theta}^2 - 2\dot{q}_1 q_2\dot{\theta} + 2q_1\dot{q}_2\dot{\theta}$$

The kinetic energy T is given by:

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$\therefore T = \frac{1}{2}m\left[\dot{q}_1^2+\dot{q}_2^2+(q_1^2+q_2^2)\dot{\theta}^2-2\dot{q}_1q_2\dot{\theta}+2q_1\dot{q}_2\dot{\theta}\right]$$

$$T = \frac{1}{2}m\left[(\dot{q}_1^2+\dot{q}_2^2)+2(q_1\dot{q}_2\dot{\theta}-\dot{q}_1q_2\dot{\theta})+(q_1^2+q_2^2)\dot{\theta}^2\right] \quad -(1)$$

The potential energy V is given by:

$$V = -\frac{1}{2}m\left[\omega_1^2x^2+\omega_2^2y^2\right]+kxy$$

$$\therefore V = -\frac{1}{2}m\left[\omega_1^2(q_1\cos\theta-q_2\sin\theta)^2+\omega_2^2(q_1\sin\theta+q_2\cos\theta)^2\right] + k(q_1\cos\theta-q_2\sin\theta)(q_1\sin\theta+q_2\cos\theta)$$

$$V = -\frac{1}{2}m\left[\omega_1^2(q_1^2\cos^2\theta+q_2^2\sin^2\theta)+\omega_2^2(q_1^2\sin^2\theta+q_2^2\cos^2\theta)\right] + k(q_1^2-q_2^2)\sin\theta\cos\theta \quad -(2)$$ (ignoring q_1q_2 product terms)

Given Lagrangian equation is

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)-\frac{1}{2}m(\omega_1^2x^2+\omega_2^2y^2) + kxy$$

i.e. $L = T - V$

$$\therefore L = \frac{1}{2}m\left[(\dot{q}_1^2+\dot{q}_2^2) + 2(q_1\dot{q}_2\dot{\theta} - \dot{q}_1 q_2\dot{\theta} + (q_1^2+q_2^2)\dot{\theta}\right]$$
$$+\frac{1}{2}m\left[\omega_1^2(q_1^2\cos^2\theta + q_2^2\sin^2\theta) + \omega_2^2(q_1^2\sin^2\theta + q_2^2\cos^2\theta)\right]$$
$$-k(q_1^2-q_2^2)\sin\theta\cos\theta \qquad \text{from (1) \& (2)}$$

$$\frac{\partial L}{\partial q_1} = \frac{1}{2}m[2\dot{q}_2\dot{\theta} + 2q_1\dot{\theta}]$$
$$+\frac{1}{2}m[2q_1\omega_1^2\cos^2\theta + 2q_1\omega_2^2\sin^2\theta]$$
$$-k(2q_1)\sin\theta\cos\theta$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{1}{2}m[2\dot{q}_1 - 2q_2\dot{\theta}]$$

$$\therefore \frac{\partial L}{\partial \dot{q}_1} = m\dot{q}_1 - q_2\dot{\theta}$$

We know that Lagrange's equations for two generalised co-ordinates q_1, q_2 are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0 \quad -(3)$$

and
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = 0 \quad -(4)$$

$$(3) \Rightarrow \frac{d}{dt}(m\dot{q}_1 - q_2\dot{\theta}) - \frac{1}{2}m[2\dot{q}_2\dot{\theta} + 2q_1\dot{\theta}]$$
$$-\frac{1}{2}m[2q_1\omega_1^2\cos^2\theta + 2q_1\omega_2^2\sin^2\theta]$$
$$+k(2q_1)\sin\theta\cos\theta = 0$$

$$m\ddot{q}_1 - q_2\ddot{\theta} - m(\dot{q}_2\dot{\theta} + q_1\dot{\theta}) - m(q_1\omega_1^2\cos^2\theta + q_1\omega_2^2\sin^2\theta)$$
$$+2kq_1\sin\theta\cos\theta = 0 \quad -(5)$$

$$\frac{\partial L}{\partial q_2} = \frac{1}{2}m\left[-2\dot{q}_1\dot{\theta} + 2q_2\dot{\theta}\right]$$
$$+\frac{1}{2}m\left[2q_2\omega_1^2\sin^2\theta + 2q_2\omega_2^2\cos^2\theta\right]$$
$$-k(-2q_2)\sin\theta\cos\theta$$

$$\therefore \frac{\partial L}{\partial q_2} = m(-\dot{q}_1\dot{\theta} + q_2\dot{\theta}) + m(q_2\omega_1^2\sin^2\theta + q_2\omega_2^2\cos^2\theta)$$
$$+2kq_2\sin\theta\cos\theta$$

$$\frac{\partial L}{\partial \dot{q}_2} = \frac{1}{2}m\left[2\dot{q}_2 + 2q_1\dot{\theta}\right]$$

$$\therefore \frac{\partial L}{\partial \dot{q}_2} = m\dot{q}_2 + q_1\dot{\theta}$$

$$④ \Rightarrow \frac{d}{dt}(m\dot{q}_2 + q_1\dot{\theta}) - m(q_2\dot{\theta} - \dot{q}_1\dot{\theta})$$
$$-m(q_2\omega_1^2\sin^2\theta + q_2\omega_2^2\cos^2\theta)$$
$$-2kq_2\sin\theta\cos\theta = 0$$

$$m\ddot{q}_2 + q_1\ddot{\theta} - m(q_2\dot{\theta} - \dot{q}_1\dot{\theta}) - m(q_2\omega_1^2\sin^2\theta + q_2\omega_2^2\cos^2\theta)$$
$$-2kq_2\sin\theta\cos\theta = 0 \qquad — ⑥$$

Equations ⑤ & ⑥ are Lagrange's equations w.r.t. q_1 and q_2 independent of parameter θ.

Q.7 a) i) Find the conjuctive normal form (CNF) of the following Boolean function :- (15M)

$f(x,y,z,t) = x\cdot y\cdot z + \bar{x}\cdot y\cdot(t+\bar{z})$.

$\Rightarrow$ $f = x\cdot y\cdot z + \bar{x}\cdot y\cdot(t+\bar{z})$

$= (x\cdot y\cdot z+\bar{x})\,(x\cdot y\cdot z+y)\,(xyz+(t+\bar{z}))$ ...... distributive law

$= (x+\bar{x})(y+\bar{x})(z+\bar{x})(x+y)(y+y)(z+y)$
$(x+t+\bar{z})(y+t+\bar{z})(z+t+\bar{z})$. ...... distributive law

$= (\bar{x}+y)(\bar{x}+z)(x+y)\,y\,(z+y)(x+t+\bar{z})$
$(y+t+\bar{z})\,(z+t+\bar{z})$ ——— (1)

...... as $x+\bar{x}=1$

is require conjuctive normal form.

[Note— upsc asked conjuctive normal form & not principle conjuctive normal form.

for principle conjuctive normal form — from eqn (1)

$\bar{x}+y = \bar{x}+y+z\cdot\bar{z} = (\bar{x}+y+z)(\bar{x}+y+\bar{z})$

let

$\bar{x}+y+z = \bar{x}+y+z+t\cdot\bar{t} = (\bar{x}+y+z+t)(\bar{x}+y+z+\bar{t})$

$\bar{x}+y+\bar{z} = \bar{x}+y+\bar{z}+t\cdot\bar{t} = (\bar{x}+y+\bar{z}+t)\cdot(\bar{x}+y+\bar{z}+\bar{t})$

$\therefore (\bar{x}+y) = (\bar{x}+y+z+t)(\bar{x}+y+z+\bar{t})$
$(\bar{x}+y+\bar{z}+t)(\bar{x}+y+\bar{z}+\bar{t})$.

in similar way proceeding for rest term in eq^n ① we will get principle conjunctive Normal form.

$$f = (x+y+z+t)(x+y+z+\bar{t})(x+y+\bar{z}+t)$$
$$(x+y+\bar{z}+\bar{t})(x+\bar{y}+z+\bar{t})(\bar{x}+y+t+\bar{z})$$
$$(\bar{x}+y+z+t)(\bar{x}+y+z+\bar{t})(\bar{x}+y+\bar{z}+\bar{t})]$$

Q.7 a ii) Express the Boolean function

$$f(x,y,z) = x + (\overline{\bar{x}\cdot\bar{y}+\bar{x}\cdot z}) + z$$

in disjunctive normal form (DNF) and construct the truth table for the function. [15M]

$\Rightarrow$ $f = x + \overline{\bar{x}\cdot\bar{y}+\bar{x}\cdot z} + z$

$= x + (x+y)\cdot(x+\bar{z}) + z$

$= x + x\cdot x + x\cdot\bar{z} + y\cdot x + y\cdot\bar{z} + z$

$= x(1+y) + x(1+\bar{z}) + (y+z)(\bar{z}+z)$

$= x + x + y + z$ .... $(1+y = 1 = 1+z = z+\bar{z})$

$= x+y+z$ .... ①

is require disjunctive normal form.

[for principle disjunctive normal form

$x = x(y+y') = xy+xy'$

$(xy+xy')\cdot(z+z') = xyz + xyz' + xy'z + xy'z'$ —— ②

same way above for y & z,

$$y = xyz + x'yz + xyz' + x'yz' \quad \ldots\ldots (3)$$

$$z = xyz + x'yz + xy'z + x'y'z \quad \ldots\ldots (4)$$

$\therefore$ from (2), (3) & (4) eqn (1) become

$$f = xyz + x'yz + xy'z + xyz' + x'y'z + x'yz' + xy'z' \quad \ldots\ldots (5)$$

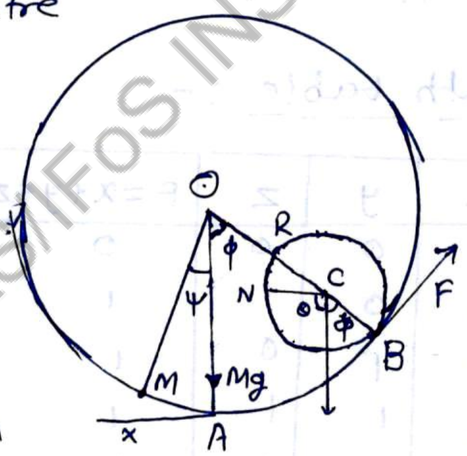
is require principle distinctive normal form.

Truth table :-

x	y	z	$F = x+y+z$	F in eqn (5)
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Q. 7 b) A perfectly rough ball is at rest within a hollow cylindrical roller. The roller is drawn along a level path with uniform velocity V. Let a and b be the radii of the ball and the roller respectively. If $V^2 > \frac{27}{7} g(b-a)$, then show that the ball will roll completely round the inside of the roller.

Soln: Let O be the centre of the roller and C the centre of the sphericalt ball moving inside the cylindrical roller.



Let CN be the radius of the ball which was vertical when it was in its lowest position.

When the roller has moved through a distance x, let CN have turned through an angle θ. The line joining the centre makes an angle φ with the vertical and the ball has turned through an angle θ. As there is no sliding,

$$\text{arc } BM = \text{arc } BN$$

i.e. $$b(\phi+\psi) = a(\theta+\phi)$$

or $$(b-a)\phi = a\theta - b\psi \quad \text{— (1)}$$

Again the velocity of the roller is constant

i.e. $$\dot{x} = b\dot{\psi} = V.$$

Then $\ddot{x} = b\ddot{\psi} = 0$ — (2)

Let R and F be the normal reaction and friction. As C describes a circle of radius $(b-a)$ about O, so accelerations along CO and perpendicular to CO are $(b-a)\dot{\phi}^2$ and $(b-a)\ddot{\phi}$ respectively.

Thus equation of motion are

$$m(b-a)\dot{\phi}^2 = R \; mg\cos\phi \quad — (3)$$

$$m(b-a)\ddot{\phi} = F - mg\sin\phi \quad — (4)$$

and $$m\frac{2a^2}{5}\ddot{\theta} = -F.a, \quad — (5)$$

eliminating F between (4) and (5), we get

$$(b-a)\ddot{\phi} = -\frac{2a}{5}\ddot{\theta} - g\sin\phi$$

or $$(b-a)\ddot{\phi} + \frac{2}{5}(b-a)\ddot{\phi} = -g\sin\phi$$

[since $(b-a)\ddot{\phi} = a\ddot{\theta} - a\ddot{\psi} = a\ddot{\theta}$ by virtue of (2)]

or $$\frac{7}{5}(b-a)\ddot{\phi} = -g\sin\phi \quad — (6)$$

Integrating it, we get

$$\frac{7}{5}(b-a)\dot{\phi} = 2g\cos\phi + A \quad — (7)$$

Initially the velocity of the C.G. is

$$\dot{x} + (b-a)\dot{\phi} = 0$$

i.e. $$(b-a)\dot{\phi} = -\dot{x} = -V$$

$$\therefore A = \frac{7V^2}{5(b-a)} - 2g$$

Candidates must not write on this margin

Hence the equation ⑦ gives

$$\frac{7}{5}(b-a)\dot{\phi}^2 = -2g(1-\cos\phi) + \frac{7v^2}{5(b-a)} \quad -⑧$$

Substituting for $\dot{\phi}^2$ from ⑧ in ③, we get

$$\frac{R}{m} = g\cos\phi + \frac{v^2}{b-a} - \frac{10}{7}(-\cos\phi)$$

$$= \frac{1}{7}\left(17g\cos\phi - 10g + \frac{7v^2}{b-a}\right)$$

The necessary condition that the ball should roll completely round the fixed cylinder is that R is positive when $\phi = \pi$, and if R is positive in this position, when it will be positive in all positions.

Hence $\left\{\frac{7v^2}{b-a} - 10 + 17g\cos\phi\right\}_{\phi=\pi} > 0$

or $\frac{7v^2}{b-a} > 27g$ or $v^2 > \frac{27g(b-a)}{7}$

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Q. 7. c) Solve the partial differential equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t>0$$

subject to the conditions

$u(0,t) = 0$, $u(L,t) = 0$, $t>0$

$u(x,0) = x$ $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 1$ $0 < x < L$

[20M]

Sol^n →

one dimensional wave equation.

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{------} \quad (1)$$

$u(x,t)$ is deflection of the string.

boundary condition $x=0$ & $x=L$

$u(0,t) = 0$ & $u(L,t) = 0$

initial deflection $u(x,0) = f(x)$ and

initial velocity $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$

then solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[E_n \cos\left(\frac{n\pi a t}{L}\right) + F_n \sin\left(\frac{n\pi a t}{L}\right)\right] \sin \frac{n\pi x}{L} \quad (A)$$

where $E_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$ &

$$F_n = \frac{2}{n\pi a}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Given $f(x) = x \quad g(x) = 1$

i) $$E_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left[-x\cos\left(\frac{n\pi x}{L}\right)\cdot\frac{L}{n\pi} + \sin\left(\frac{n\pi x}{L}\right)\cdot\frac{L^2}{n^2\pi^2}\right]_0^L$$

$$= \frac{2}{L}\left[\frac{-L^2}{n\pi}\cos n\pi + \frac{L^2}{n^2\pi^2}\sin n\pi\right]$$

$$= \frac{-2L}{n\pi}\cos n\pi$$

$$= \frac{2L}{n\pi}\begin{cases} -1 & n \text{ is even} \\ +1 & n \text{ is odd}\end{cases}$$

ii) $$F_n = \frac{2}{n\pi a}\int_0^L 1\cdot\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{n\pi a}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{-2}{n^2\pi^2 a}\left[+\cos n\pi - 1\right]$$

$$= \frac{2}{n^2\pi^2 a}\left[1 - \cos n\pi\right]$$

$$= \begin{cases} \frac{4}{n^2\pi^2 a} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even.}\end{cases}$$

Put value of E_n & F_n in eq^n (A)

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{-2L}{n\pi} \cos\left(\frac{n\pi at}{L}\right)\right] . \sin \frac{n\pi x}{L} \quad \text{——} \quad (B)$$

when n is even

&

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2L}{n\pi} \cos\left(\frac{n\pi at}{L}\right) + \frac{4}{n^2\pi^2 a} \sin\left(\frac{n\pi at}{L}\right)\right]$$

$$\times \sin\left(\frac{n\pi x}{L}\right) \quad \text{-----} \quad (C)$$

when n is odd.

Hence eq^n (B) & eq^n (C) are required solution of given PDE. when n is even & n is odd respectively.

Q. 8. a Reduce the partial differential equation

$$\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\left(1+\frac{1}{x}\right) + \frac{z}{x} = 0$$

to canonical form. [15 marks]

Sol^n: Writing the given eq^n as

$$t - s + p - q\left(1+\frac{1}{x}\right) + \frac{z}{x} = 0$$

$$\Rightarrow \quad 0\cdot r - s + t + p - q\left(1+\frac{1}{x}\right) + \frac{z}{x} = 0 \quad \text{—(1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p) = 0$

here $R = 0$, $S = -1$ & $T = 1$

Hence $S^2 - 4RT = 1 > 0 \Rightarrow$ given eq^n is hyperbolic

The λ- quadratic equation $R\lambda^2 + S\lambda + T = 0$

reduces to $-\lambda + 1 = 0$ giving $\lambda = 1$.

Hence the corresponding characteristic equation

$\frac{dy}{dx} + \lambda = 0$ yields $\frac{dy}{dx} + 1 = 0$ or

$dx + dy = 0$

Integrating it, $x + y = c$,

where c is arbitrary constant.

choose $u = x+y$ and $v = x$ — (2)

we have chosen $v=x$ in such a manner that u and v are independent verified

as $J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$

$\Rightarrow$ u and v are independent functions.

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$

$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$ using (2) — (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}$ using (2) — (4)

from (4) $\Rightarrow \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial u}$ — (5)

$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$ — (6)

using (3) & (5)

and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$

using (5)

or $t = \frac{\partial^2 z}{\partial u^2}$ — (7)

using ②, ③, ④, ⑥ & ⑦

eqn ① reduces to

$$-\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u \partial v}\right)+\frac{\partial^2 z}{\partial u^2}+\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\left(1+\frac{1}{v}\right)+\frac{z}{v}=0$$

or,

$$\frac{\partial^2 z}{\partial u \partial v}-\frac{\partial z}{\partial v}+\frac{1}{v}\cdot\frac{\partial z}{\partial u}-\frac{z}{v}=0$$

which is required Canonical form.

8 (b). compute a root of the equation $\log_{10}(2x+1) - x^2 + 3 = 0$, in the interval $[0, 3]$, by Regula-Falsi method, correct to 5 decimal places.

sol^n →

$$f(x) = \log_{10}(2x+1) - x^2 + 3$$

$f(0) = 3 \quad f(3) = -5.1549$

$f(0) \cdot f(3) < 0$, hence we will get one root lies in $[0, 3]$.

In order to achieved 5 decimal in minimum no. of step we need to choose value of x at which $f(x)$ is close to zero.

$x_0 = 1.9 \quad f(x_0) = 0.071241 > 0$

$x_1 = 1.95 \quad f(x_1) = -0.1123 < 0$

hence choose $x_0 = 1.9$ & $x_1 = 1.95$ as initial value. $[f(x_0) \cdot f(x_1) < 0]$.

we know the regula falsi method formula.

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

Iteration 1 – $x_0 = 1.9$ $f(x_0) = 0.071241$

$x_1 = 1.95$ $f(x_1) = -0.1123$

$$x_2 = 1.9 - \frac{1.95 - 1.90}{(-0.1123 - 0.071241)} \cdot (0.071241)$$

$$\boxed{x_2 = 1.919407}$$

$f(x_2) = 6.15696 \times 10^{-4} > 0$. Hence replace x_0 by x_2

Iteration 02 –

$x_0 = 1.919407$ ..... (new)

$x_1 = 1.95$

$$\boxed{x_2 = 1.919573815}$$

Iteration 03 -

$x_2 = 1.919573815$ $f(x_2) = 5.2417 \times 10^{-6}$

$f(x_2) > 0$ hence replace x_0 by x_2.

$x_0 = 1.919573815$ ..... (new)

$x_1 = 1.95$

$$\boxed{x_2 = 1.91957552}$$

Iteration 4

$x_0 = 1.91957552$

$x_1 = 1.95$

$$\boxed{x_2 = 1.91957524}$$

Hence require root by regula falsi method is 1.91957524.

Q. 8.c Determine under what conditions the velocity field $u = c(x^2-y^2)$, $v = -2cxy$, $w=0$ is a solution to the Navier-Stokes momentum eq^n. Assuming that the conditions are met, determine the resulting pressure distribution, when z is up and external body forces are $B_x = 0 = B_y$, $B_z = -g$.

[20 Marks]

Sol^n:

w.k.t

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho \frac{du}{dt}$$

here $g_x = B_x$

$$\therefore \rho(0) - \frac{\partial p}{\partial x} + \mu(2c-2c) = 2c^2\rho(x^3+xy^2) \quad \text{—} (1)$$

when compressibility is significant, additional small terms arise containing the element volume expansion rate and a second coefficient of viscosity.

$$\therefore \rho(0) - \frac{\partial p}{\partial y} + \mu(0) = 2c^2\rho(x^2y+y^3) \quad \text{—} (2)$$

$$\rho(-g) - \frac{\partial p}{\partial z} + \mu(0) = 0 \quad \text{—} (3)$$

The viscous term vanishes identically (although μ is not zero). Equation (3) can be integrated partially to obtain

$$p = -\rho g z + f_1(x, y) \quad \text{--- (4)}$$

i.e the pressure is hydrostatic in the Z-direction.

As the flow is two Dimensional ($w=0$)

Differentiating equation (1) w.r.t y

$$\frac{\partial^2 p}{\partial x \partial y} = -4c^2 \rho x y \quad \text{--- (5)}$$

Differentiating eq. (2) w.r.t x

$$\frac{\partial^2 p}{\partial x \partial y} = -\frac{\partial}{\partial x}\left[2c^2 \rho (x^2 y + y^3)\right] = -4c^2 \rho x y$$

--- (6)

Since these are identical, the given velocity field is an exact solution to the Navier-Stokes equation.

To find pressure distribution, substitute Eq. (4) into Eq. (1) and (2) to find $f_1(x, y)$

$$\frac{\partial f_1}{\partial x} = -2c^2\rho(x^3+xy^2) \quad \text{--- } (7)$$

$$\frac{\partial f_1}{\partial y} = -2c^2\rho(x^2y+y^3) \quad \text{--- } (8)$$

Integrate Eq. (7) partially w.r.t x

$$\therefore f_1 = -\frac{1}{2}c^2\rho(x^4+2x^2y^2) + f_2(y) \quad \text{--- } (9)$$

Differentiate this w.r.t y and Compare with Eq. (8)

$$\frac{\partial f_1}{\partial y} = -2c^2\rho x^2 y + f_2'(y) \quad \text{--- } (10)$$

Comparing (8) & (10)

$$f_2'(y) = -2c^2\rho y^3$$

$$\text{or, } f_2(y) = -\frac{1}{2}c^2\rho y^4 + C \quad \text{--- } (11)$$

where C is constant.

Combining Eq: (4), (9) and (11) to give the complete expression for pressure distribution

$$p(x,y,z) = -\rho g z - \frac{1}{2}c^2\rho(x^4+y^4+2x^2y^2) + C$$

This is the desired solution.

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