CALCULUS

Syllabus :- Paper I, Section A. (70-90 marks)

[Real numbers, functions of a real variable, limits, continuity differentiability, mean value theorems, Taylor's theorem with remainders Indeterminate form, maxima & minima, Asymptotes | Curve tracing |,] functions of two & three variables, limits, continuity partial deravatives, maxima & minima, lagrange's method of multipliers, Jacobian, Riemann definition of Definite integrals, indefinite integrals Infinite & improper integrals, Double & Triple integrals (Evaluation techniques only), Areas & surfaces & Volume

Do in real analysis

⇒ Limits :-

$\lim_{x \to a} f(x) = l \quad \because [x \neq a]$

for each $\epsilon > 0$, $\exists \ \delta > 0$

Such that $|f(x) - l| < \epsilon$ whenever $0 < |x-a| < \delta$

nbd
$|x-a| < \delta$ adding a
$-\delta < x-a < \delta$
$x \in (a-\delta, a+\delta)$

Deleted nbd: $\{x \in (a-\delta, a) \cup (a, a+\delta)$

$-\epsilon < f(x) - l < \epsilon$

$l - \epsilon < f(x) < l + \epsilon$

$\Rightarrow \ f(x) \in (l-\epsilon, l+\epsilon)$

$\boxed{\delta \text{ is function of } \epsilon}$

(1) S.T $\lim_{x \to a} \frac{x^2 - a^2}{x-a} = 2a$

(Calculus method)

Sol: $\lim_{x \to a} \frac{x^2-a^2}{x-a} = \lim_{x \to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{(x-a)}}$

$= \lim_{x \to a} (x+a)$

$= 2a$

Ans

proved

Definition method : (Real analysis method)

$f(x) = \dfrac{x^2 - a^2}{x - a}$ $\quad x \neq a$

$l = 2a$

$|f(x) - l| < \epsilon$ when $0 < |x - a| < \delta$

$\Rightarrow \left| \dfrac{x^2 - a^2}{x - a} - 2a \right| < \epsilon$

$\Rightarrow \left| \dfrac{\cancel{(x-a)}(x+a)}{\cancel{(x-a)}} - 2a \right| < \epsilon$

$= |x + a - 2a| < \epsilon$

$\Rightarrow |x - a| < \underline{\epsilon}$ whenever $0 < |x - a| < \underline{\delta}$

by choosing $\delta = \epsilon$

then for each $\epsilon > 0$, $\exists\ \delta > 0$

such that $|f(x) - l| < \epsilon$

whenever $0 < |x - a| < \delta$

$\lim_{x \to a} f(x) = l$

$\lim_{x \to a} \dfrac{x^2 - a^2}{x - a} = 2a$

$$\boxed{a^3 - b^3 = (a-b)(a^2+ab+b^2)}$$

2) S.T $\lim_{x \to 0} x(\sin 1/x) = 0$

Soln: calculus

$$\lim_{x \to 0} x \sin(1/x)$$

$$= \lim_{x \to 0} x \cdot \lim_{x \to 0} \sin(1/x)$$

$\because (1/x = 1/0 = \infty)$

$$= 0 \cdot \text{finite value}$$

$$= 0 \quad \text{RHS}$$

Real analysis method

$f(x) = x \sin(1/x)$ $\quad l = 0, \quad a = 0$

$|f(x) - l| < \epsilon$ whenever $0 < |x-a| < \delta$

$$= |x \sin(1/x) - 0| < \epsilon \quad \text{—''—}$$

$$\Rightarrow |x \sin(1/x)| < \epsilon \quad \text{—''—}$$

$$\Rightarrow |x| \, |\sin 1/x| < \epsilon \quad \text{—''—}$$

$$\Rightarrow |x| \cdot 1 < \epsilon$$

$\boxed{|\sin 1/x| \leq 1}$

$|x| < \epsilon \quad , \quad 0 < |x| < \delta$

by choosing $\delta = \epsilon$ then

for each $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$|f(x) - l| < \epsilon$ whenever $0 < |x-a| < \delta$

$\therefore \lim_{x \to a} f(x) = l$

$\therefore \lim_{x \to 0} x \sin(1/x) = 0$ Proved

$\Rightarrow$ Some imp points of limits

1. Uniqueness of a limit

2. $\lim_{x \to a} f(x) = l$, $\lim_{x \to a} g(x) = m$

(a) $\lim_{x \to a} f(x) \pm g(x) = l \pm m$

(b) $\lim_{x \to a} f(x) \cdot g(x) = lm$

(c) $\lim_{x \to a} \dfrac{f(x)}{g(x)} \; (g(x) \neq 0) = \dfrac{l}{m} \; (m \neq 0)$

3. • $\lim_{x \to a} f(x) = l$ $f(x) \geq 0$ $\forall x \in D$ (domain)

$\Rightarrow l \geq 0$

• $f(x) > 0$ $\forall x \in D$

$\Rightarrow l \geq 0$

'Real Analysis'

Paper 2: section 'A'

Syllabus:-

Real number system as an Ordered field with the least upper bound property: Sequences, the limit of a sequences, Cauchy sequence, completeness of real line, Series & it's convergence, absolute & conditional convergence of series of real & complex terms, rearrangements of series, Continuity & uniform continuity of function.

Properties of Continuous function on compact sets, Riemann integral, improper integrals, Fundamental theorems of integral calculus, Uniform convergence, continuity, differentiability & integrability for sequences & series of functions, partial derivatives of functions of several (2 or 3) variables, maxima & minima.

Fields & Ordered Fields

$a+b,\ a-b,\ a.b\ ,\ a\div b\ \ (b\neq 0)\ \in S$

$a,b \in S$

Rational no., Real no., Complex no. are fields.

Ordered Structure :-

(i) $a<b$ and $b<c \Rightarrow a<c$ (Transitive relation)

(ii) $a<b \Rightarrow a+c < b+c$

(iii) $a<b \Rightarrow ac < bc \quad (c>0)$

'Rational' & '~~Irrational~~' 'Real' are Ordered fields.

Real Number As a Complete Ordered Field :-

Set

$a < b$

Closed interval $[a, b] = \{x \mid a \le x \le b\}$

$[a, b[$ Semiclosed or semi open interval $\begin{cases} [a, b) = \{x \mid a \le x < b\} \\ (a, b] = \{x \mid a < x \le b\} \end{cases}$

$]a, b]$

open interval $(a, b) = \{x \mid a < x < b\}$

$\downarrow$

$]a, b[$

- Set bounded above :- 'S' is a subset of Real number ($S \subseteq R$) we say that 'S' is bounded above (bdd). if there exists a real number 'b' not necessarily member of 'S' such that $x \le b \ \forall \ x \in S$ and the number 'b' is called Upperbound of the set 'S'.

Eg:- $S = \{1, 2, 3, 4\}$ $\qquad$ $b + \epsilon$ is always upper bound

$x \le 4 \ \forall \ x \in S$

So this is set bounded above & 4 is upperbound so all the number greater than 4 also form upperbound

- Not bounded above set :- $\nexists$ (there doesn't exists) $b \in R$ such that $x \le b \ \forall \ x \in S$

- Bounded Below Set :- $S \subseteq R$, 'S' is bounded below set if $\exists \ a \in R$ not necessarily a member of S

$a \le x \ \forall \ x \in S$

'a' is lower bound

$a - \epsilon \rightarrow$ Is always lower bound

Eg:- ① $S = \{1, 2, 3, 4\}$

'1' is lower bound

$1 \leq x \quad \forall \; x \in S$

② $S = N = \{1, 2, 3 \ldots \}$

$0 < x \quad \forall \; x \in S$

$1 \leq x \quad \forall \; x \in S$

- Bounded Set :- A set is Both bounded below as well as bounded above $a \leq x \leq b \quad \forall \; x \in S$

- Least Upperbound (l.u.b) or Supremum or Sup :-
  The least upperbound element Supremum
  Eg:- $S = \{1, 2, 3, 4\}$
  Here 4 is l.u.b or Sup

- Greatest lowerbound (g.l.b) or Infimum or Inf :-
  The greatest lowerbound element is Infimum
  Eg:- $S = \{1, 2, 3, 4\}$
  Here '1' is g.l.b or Inf

- Greatest & Smallest Member of a Set :-
  The element must contain in a 'Set'

① $S = [1, 2, 3, 4]$

1 → Smallest member, 4 → Greatest member

② $S = [3, 4)$

3 → Smallest member. so there is no Greatest member

③ $(3, 4]$

No smallest number, 4 is greatest Number.

- Set of all the upper bound elements is Bounded Below Set.
- Set of all the lower bound elements is Bounded Above Set.

Set	Bounded Below	Bound above	Bounded	Sup	Inf	Greatest	Smallest
$S = N = \{1, 2, 3, 4 \ldots\}$	✓	×	×	-	1	Doesn't exist	1
$S = I^{-1} = \{-1, -2, -3 \ldots\}$	×	✓	×	-1	-	-1	DNE
$S = J = \{0, \pm 1, \pm 2, \pm 3 \ldots\}$	×	×	×	-	-	DNE	DNE
$I = Q = Q^c = R$ Integers = Rational = Irrational = Real							
(A) $S = (a, b)$	✓	✓	✓	b	a	DNE	a
(5) $S = [a, b)$	✓	✓	✓	b	a	DNE	a
(6) $S = (a, b]$	✓	✓	✓	b	a	b	DNE
(7) $S = [a, b]$	✓	✓	✓	b	a	b	a
$S = \{a\}$	✓	✓	✓	a	a	a	a
$S = \{1, 2, 3, 4\}$	✓	✓	✓	4	1	4	1
(10) $S = \{\frac{1}{n} \mid n \in N\}$	✓	✓	✓	1	0	1	DNE
$S = \{(-1)^n \, 1/n \mid n \in N\}$ $-1 \le x \le 1/2$	✓	✓	✓	1/2	-1	1/2	-1