Modern Algebra

Paper 2

Section A :- (50 to 70) marks (2013 to 2021 PYQ)

Syllabus

Groups, Subgroups, Cyclic groups, Cosets, Lagrange's theorem, Normal subgroup, Quotient groups, Homomorphism of groups, basic isomorphism theorem, permutation groups, Cayley's theorem; Rings, Subrings, ideals, homomorphism of rings, Integral domain, principal ideal domain, Euclidean domains & unique factorization domains, Fields, Quotient fields

Groups :-

$G \neq \phi$ (not empty set)

1. Closure property : $\forall a,b \in G \quad a*b \in G$
2. Associative property : $(a*b)*c = a*(b*c)$
   $\forall a,b,c \in G$

If G follows both closure & associative property then it is Semi group

If only closure property follows → Algebraic structure

3. Existence of identity :-
   $\forall a \in G \quad \exists e \in G$ such that $a*e = e*a = a$

4. Existence of inverse
   $\forall a \in G \quad \exists b \in G$ such that $a*b = b*a = e$
   b is inverse of a, e is identity element.
   $(b = a^{-1})$

If it follows all 4 property than it is called Group.

5. If commutative property :- $\forall a,b \in G \quad a*b = b*a$
   If it follows all 5 property called as abelian (commutative) group.

· If number of G is finite, called finite group
" ——— infinite ——— infinite group.
If it is infinite along with abelian group called as infinite abelian group.

• Integers, Rational, Real & Complex number s/m → Infinite Abelian group

$(I,+)$ $(Q,+)$ $(R,+)$, $(C,+)$

• Q_0, R_0, C_0 are infinite abelian groups

$Q_0 = Q - \{0\}$
(All numbers except zero)

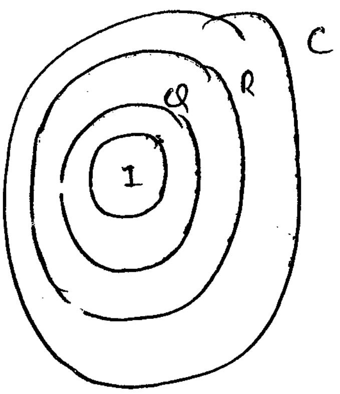
$R_0 = R - \{0\}$

$C_0 = C - \{0\}$

$(Q,\cdot)$
$(R,\cdot)$
$(C,\cdot)$

$(I,\cdot)$ → is not group

$(Q_0,+)$, $(R_0,+)$, $(C_0,+)$ → is not a group



$I \subset Q \subset R \subset C$

⇒ Some properties and groups :-

* The identity element in a group(e) is unique
* The inverse of each element of a group is unique
* $(a^{-1})^{-1} = a$
* $(ab)^{-1} = b^{-1}a^{-1}$

note: If G is abelion group than $(ab)^{-1} = a^{-1}b^{-1}$

* Cancellation laws hold true in a group

that is i.e $a, b, c \in G$

$ab = ac \Rightarrow b = c$ (Left cancellation law)

$ba = ca \Rightarrow b = c$ (Right cancellation law)

* If a, b are any two elements of group $(a, b \in G)$ then $ax = b$ & $ya = b$ have unique solutions

The soln is $x = a^{-1}b$ & $y = ba^{-1}$

Definition of a group based on left (Right) axiams :-

1) Closure property	1) —" ______
2) Associative property	2) —" ______
3) Existence of left identity $e * a = a$	3) Existence of right identity $a * e = a$
4) Existence of left inverse $b * a = e$	4) Existence of right inverse $a * b = e$
⇓ Show in this line	⇓ Show in this line

Anyone

(1) The set I of all integers the operation defined by $*$ as $a*b = a+b+1$ PST $(I, *)$ is an Abelian group.

Soln

(i) Closure:

let $a, b \in I$, $1 \in I$

$a+b+1 \in I$

$a * b \in I$

$\forall\ a, b \in I \quad a*b \in I$

Closure is true

(ii) Associative

$a, b, c \in I$

$(a * b)*c = a*(b*c)$

$LHS = (a*b)*c = (a+b+1)*c = a+b+1+c+1 = a+b+c+2$

$RHS = a*(b*c) = a*(b+c+1) = a+1+b+c+1 = a+b+c+2$

$LHS = RHS$

So associative is true

(iii) Existence of left identity

let e be the left identity

$e*a = a$

$e+\cancel{a}+1 = \cancel{a}$

$e = -1 \in I$

$-1*a = -1+a+1 = a$

Therefore -1 is the left identity

Linear Algebra :-

Paper 1 , Section A , 60-70 marks.

Syllabus.

Vector spaces over R & C, linear dependence & Independence, Subspaces, bases, dimension ;
Linear transformations, rank & nullity, matrix of a linear transformation. [Algebra of matrices, Row & column reduction, Echelon form, Congruence & Similarity, Rank of a matrix, Inverse of a matrix, Solution of s/m of linear eqn....... -- matrices.]

Vector Spaces :-

Field :- $(F, +, \cdot)$

F is any set but not empty set

For field F contains atleast 2 elements

$+, \cdot \rightarrow$ addition & $\times^{n}$ operation.

properties to satisfy field

F_1 : $a+b = b+a \quad \forall\, a, b \in F$ (addition is commutative)

F_2 : $(a+b)+c = a+(b+c) \quad \forall\, a, b, c \in F$ (addition is associative)

F_3 : $\forall\, a \in F \;\exists\; 0 \in F$ (zero element) such that $a+0 = a$

F_4 : $\forall\, a \in F \;\exists\; -a \in F$ such that $a+(-a) = 0$ (zero element)

F_5 : $a \cdot b = b \cdot a \quad \forall\, a, b \in F$ (multiplication is commutative)

F_6 : $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall\, a, b, c \in F$ ($\times^{n}$ is Associative)

F_7 : $\forall\, a \in F \;\exists\; 1 \in F$ (unity element) such that $a \cdot 1 = a$

F_8 : $\forall\, a \neq 0 \in F \;\exists\; a^{-1} \in F$ such that $a \cdot a^{-1} = 1$ (Unity element)

F_9 : $\forall\, a, b, c \in F \quad a \cdot (b+c) = a \cdot b + a \cdot c$ (Distributive property)

(1) Natural numbers $N = \{1, 2, 3 \ldots\}$

It satisfies $(N, +)$, $(N, \cdot)$, $(N, +, \cdot)$

but it doesn't satisfy F_3 so it is not field

(2) Integers

$I = \{0, \pm 1, \pm 2, \ldots\}$

Follows $F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_9$

but not follows F_8, so it is not a field

(3) Rational numbers

$Q = \{p/q \mid q \neq 0, p \in I\}$

It follows all F_1 to F_9

So it is field $\Rightarrow$ Rational field.

(4) Real numbers

$R = Q \cup Q^*$

Q = Rational number $\quad Q^*$ = Irrational number

So it is field [follow f_1 to F_9]

(5) Complex number

$C = \{a + ib \mid a, b \in R, i^2 = -1\}$

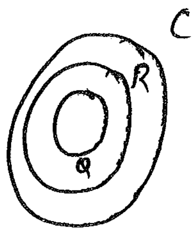
So it is field follow f_1 to F_9.

- The fields are Rational, Real & Complex numbers.

Subfield :- K is a subfield of F.

A field inside the field.

$(K, +, \cdot)$



⇒ **Vector space or Linear space :-**

V(F) is said when,

(i) $\alpha + \beta \in V \quad \forall \alpha, \beta \in V$

(ii) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$
$\forall \alpha, \beta, \gamma \in V$

(iii) $\forall \alpha \in V \ \exists \ 0 \in V$ such that
$\alpha + 0 = 0$

(iv) $\forall \alpha \in V \quad \exists \ -\alpha \in V$ such that
$\alpha + (-\alpha) = 0$

(v) $\alpha + \beta = \beta + \alpha \quad \forall \alpha, \beta \in V$

(i). Internal composition

V or A is set, * is any operation: $a, b \in A \Rightarrow a * b \in A$ | $\alpha * \beta$

(ii) External composition

$(F, +, \cdot)$ $\boxed{a, b, c \ldots}$

$a \odot \alpha = a\alpha$ is external

Elements inside set A / V are referred as Vectors

Inside elements of the field called as Scalars

I $(V, +) \rightarrow$ Abelian group or commutative group

II : $a\alpha \in V \quad \forall a \in F, \ \forall \alpha \in V$

III :-

(i) $a(\alpha+\beta) = a\alpha + a\beta \quad \forall a \in F, \ \forall \alpha, \beta \in V.$

(ii) $(a+b)\alpha = a\alpha + b\alpha \quad \forall a, b \in F, \quad \forall \alpha \in V$

(iii) $(ab)\alpha = a(b\alpha) \quad \forall a, b \in F, \quad \forall \alpha \in V$.

(iv) $1\alpha = \alpha, \quad \forall \alpha \in V, \quad 1 \in F$.

Internal composition + External composition.

Q(Q) ✓	R(Q) ✓	C(Q) ✓
Q(R)	R(R) ✓	C(R) ✓
Q(C)	R(C)	C(C) ✓

Q(Q), R(Q), R(R), C(Q), C(R), C(C) ⇒ Forms vector space as it satisfies all I, II, III condn.

Always subfield is | forms Vector space.

F(K) will be always form vector space

F is field
K is subfield

MATRICES

Syllabus :

Algebra of Matrices, Row & Column reduction, Echelon form, Congruence, and Similarity ; Rank of a matrix, Inverse of a matrix, Soln of s/m of Linear equations, Eigen values & Eigen vectors, Characteristic polynomials, Cayley Hamilton theorem, Symmetric, Skew symmetric, Hermitian, Skew Hermitian, Orthogonal & Unitary matrices and their eigen values.

Rank of a matrix :-

(1) Submatrix of a Matrix :-

Suppose A is a any matrix of type $m \times n$ than the matrix obtained by leaving some rows & columns from A is called submatrix of 'A'

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 9 \end{bmatrix}_{3\times4}$$

$$B = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \\ 6 & 5 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 5 & 4 \end{bmatrix}$$

B, C are submatrix of A

(2) Minors of a matrix :-

If A be the $m \times n$ matrices than [redacted] determinant of every square submatrix of A is called Minors of that matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$$

$1\times1 \to 6$ $\quad [1], [2], [3], [4], [5], [6]$

$2\times2 \to 3$ $\quad \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = -3, \quad \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} = -3, \quad \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = -6$

Those above are termed as minors of a matrix.

$\underbrace{-3, -3, -6}_{\text{Determinants}}$ are called 2 nd minors,

Rank of a matrix :-

The number 'r' is said to be ROM 'A' if it possesses following 2 properties

(i) There is atleast one squares of sub matrix of A of order r whose determinant not equal to zero

(ii) If the matrix A contains any squares of matrix r+1, than determinant of every square sub matrix of A of order r+1, must be / should be zero & Rank of A denoted by

$\rho(A) = r$

Ex:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$

$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = -3 \neq 0$

So $\rho(A) = 2$

Rank of null matrix = 0

Note :-

(i) A is null matrix $\Leftrightarrow \rho(A) = 0$

(ii) $A \neq 0$ than $\rho(A) > 0 \quad (\neq 0)$

(iii) $\rho(I_n) = n$, I_n = n^{th} order unit matrix

Ex:

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

$(6-8) - 2(4-1) + 3(4-0)$

$= -2 - 2(3) + 12$

$\neq 0 \quad \rho(A) = 3$

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$1(24-25) - 2(18-20) + 3(15-16)$

$\rho(A) = 2$

(c) $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \end{bmatrix}$

$3 \quad 1 \quad 2$

$3(4-4) - 1(12-12) + 2(6-6)$

$\rho(A) = 1$

(d) $A = \begin{bmatrix} 3 & 4 & 3 & 2 \\ 3 & 5 & 1 & 4 \end{bmatrix}$

$\rho(A) = 2$

ECHELON FORM :-

A matrix A said to be echelon form if

(i) Every row of A which has all entrys zeros occurs below every row which has non zero entry

(ii) The number of 0's before the 1st non zero element in a row is less than number of such zeros in the next row.

$\rho(A)$ = no of non zero rows

(1) Find the Rom $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$ [no of non zero rows]

(2) Find the Rom

$A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, ~~$R_4 \to R_4 - 2R_3$~~, ~~$R_5 \to R_5 - 3R_3$~~.

$$= \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 2 & 2 & 2 & 2 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 10R_1$, $R_5 \to R_5 - 15R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$R_4 \to R_4 - R_2$, $R_5 \to R_5 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$ [no of non zeroes rows]