ODE - Ordinary Differential Equations

Paper-I

Section - B

Syllabus :- Formulation of ODE, Equations of 1st Order & 1st Degree, Integrating factors, Orthogonal Trajectory. Equations of 1st Order but not a 1st degree,

Clairaut's Equations, Singular solⁿ

Second & higher order Linear Differential Equations with Constant Co-efficients, Complementary function, Particular integral & General Solution, 2nd Order Linear equation with variable Co-efficients, Euler Cauchy equations, Determination of complete solution when one solution is known by using method of Variation of Parameters, Laplace & Inverse laplace eqⁿ with there property, Laplace transformations of Elementary functions. Application to initial value problems for 2nd Order linear differential eqⁿ with Constant co-efficients

Formulation of Ordinary Differential Equation :-

Rule: Number of arbitrary Constants is must be equal to order of Differential eq^n

1) $x^2+y^2+2ax+2by+c=0$ (a,b,c) are constants

$\frac{d}{dx}$ $2x+2yy_1+2a+2by' = 0$

$x+yy_1+a+by_1=0$

Again d. wr to x.

$1+y_1^2+yy_2+by_2=0$ — (1)

Again wr to x

$yy_3+3y_1y_2+by_3=0$ — (2)

(1) $\times y_3$ — (2) y_2

$= y_3[1+yy_2+y_1^2+by_2] - y_2[yy_3+3y_1y_2+by_3]$

$\Rightarrow y_3+\cancel{yy_2y_3}+y_3y_1^2-\cancel{yy_2y_3}+[-3y_1y_2^2]=0$

$$\Rightarrow \boxed{y_3[1+y_1^2]+[-3y_1y_2^2]=0}$$

$$y_3(1+y_1^2)-3y_1y_2^2=0$$

If $y = c(x-c)^2$ —(1)

w.r.to x

$$y_1 = c \cdot 2(x-c)(1)$$

$$y_1 = 2c(x-c) \quad \text{—(2)}$$

Again w.r.to x

$$y_2 = 2c[1]$$

$$y_2 = 2c$$

(2) ; (1) $\quad \dfrac{y_1}{y} = \dfrac{2\cancel{c}(x-c)}{\cancel{c}(x-c)^2}$

$$x - c = \frac{2y}{y_1} \quad \text{—(3)}$$

$$x - \frac{2y}{y_1} = c$$

$$\frac{xy_1 - 2y}{y_1} = c \quad \text{—(4)}$$

$$y = \left[\frac{xy_1 - 2y}{y_1}\right]\left[\frac{2y}{y_1}\right]^2$$

$$y = \left(\frac{xy_1 - 2y}{y_1}\right)\left[\frac{4y^2}{y_1^2}\right]$$

$$\boxed{y_1^3 = 4xyy_1 - 8y^2}$$

If $y = a\sin x + b\cos x + ce^x + de^{-x}$

eliminate a, b, c, d

w.r.to x

$$y_1 = a\cos x - b\sin x + ce^x - de^{-x} \quad \text{—(1)}$$

$$y_2 = -a\sin x - b\cos x + ce^x + de^{-x}$$

$$y_3 = -a\cos x + b\sin x + ce^x - de^{-x}$$

$$y_4 = a\sin x + b\cos x + ce^x + de^{-x}$$

$y_4 = y$

$$\boxed{y_4 - y = 0}$$

4) $y = e^{mx}$

$\cancel{y_1 = e^{mx} \cdot m}$

$\cancel{y_1 = m e^{mx}}$

Apply log

$\log y = \log e^{mx}$ or $\log y = mx$ - (a)

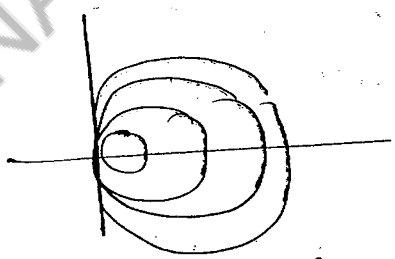
$1/y\,(y_1) = m$ - (1)

$\log y = (y_1/y)\,x$

$$\boxed{x y_1 - y \log y = 0}$$

8. Find the differential eq^n of all circles which passes through origin & whose centres on x-axis (or)

- Find the differential eq^n. of s/m of circle touching y-axis at origin (or)
- Find the differential eq^n of circle touching the given straight line at a given point.



$(-g, -f)$ $\quad r = \sqrt{g^2+f^2-c}$

$x^2+y^2+2gx+2fy+c=0$ (1)

if at x axis

$f=0$ $\quad c=0$

$$x^2+y^2+2gx=0$$

w.r.to x

$$2x+2yy_1+2g=0 \Rightarrow 2g=-2x-2yy_1$$

$$x^2+y^2+x(-2x-2yy_1)=0$$

$$\boxed{y^2-x^2-2xyy_1=0}$$

Paraboles

$$(y-k)^2=4a(x-h)$$

$y^2=4a(x+a) \rightarrow$ Confocal / Coaxial parabola

Ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{(x-a)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$\boxed{\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1} \rightarrow$ Confocal / Coaxial Ellipse

Slope point formula

$x^2=4ay \left\{ \begin{array}{l} y-y_1=m(x-x_1) \\ (at_1^2, 2at) \end{array} \right.$ | $yt=x+at^2$, $y=\frac{x}{t}+at$

$y=mx+\frac{a}{m} \rightarrow$ Tangent Eqn of Parabola.

If circle passes through origin :-

$$x^2+y^2+2gx+2fy=0 \quad (g,f)$$

PARTIAL DIFFERENTIAL EQUATIONS (PDE)

Syllabus : Paper 2 (Section B)

Family of Surfaces in 3 dimension & formulation of PDE Solns of Quasi linear PDE of 1st Order, Cauchy's Method of Characteristics.

Linear PDE of 2nd Order with constant co-efficients, canonical forms, Eqn of Vibrating string, Heat eqn, Laplace eqn & there solutions.

→ Formation of PDE

PDE :- An eqn containing one or more partial derivatives of unknown functions of 2 or more independent variables is known as PDE.

Eg:-

- $\frac{\delta z}{\delta x} + \frac{\delta z}{\delta y} = z + xy$
- $\left(\frac{\delta z}{\delta x}\right)^2 + \frac{\delta^3 z}{\delta y^3} = 2x \frac{\delta z}{\delta x}$
- $z \left(\frac{\delta z}{\delta x}\right) + \frac{\delta z}{\delta y} = x$
- $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = xyz$
- $\frac{\delta^2 z}{\delta x^2} = 1 + \left(\frac{\delta z}{\delta y}\right)^{1/2}$
- $y\left(\frac{\delta z}{\delta x}\right)^2 + \left(\frac{\delta z}{\delta y}\right)^2 = z\frac{\delta z}{\delta y}$

Order of the PDE

It is defined as the order of the highest partial derivative occuring in the PDE.

Eg: $\frac{\delta x}{\delta y} = 1$, $\frac{\delta^2 x}{\delta y^2} = 2$, $\frac{\delta^3 x}{\delta y^3} = 3$, $\left(\frac{\delta x}{\delta y}\right)^2 = 1$

Degree of PDE

The degree of the PDE is the degree of highest order derivative which occurs in it after the eq^n has been rationalized, that is made free from radicle & fractions as far as derivative are concerned

$\left(\frac{\delta x}{\delta y}\right)^1 = 1$, $\frac{\delta^2 z}{\delta x^2} = \left(1 + \frac{\delta z}{\delta y}\right)^{1/2}$

For such kind

$$\left(\frac{\delta^2 z}{\delta x^2}\right)^2 = \left[\left(1 + \frac{\delta z}{\delta y}\right)^{1/2}\right]^2$$

$$\left(\frac{\delta^2 z}{\delta x^2}\right)^2 = \left(1 + \frac{\delta z}{\delta y}\right)^1$$

Degree = 2.

Linear and Non linear PDE

A PDE is said to be linear if the dependent variable & its partial derivatives occurs only in the 1st degree & are not multiplied together.

$\underbrace{z = f(x, y)}$ Dependent variable

Independent variable

Eg: $\frac{\delta x}{\delta y} + \frac{\delta z}{\delta y} =$ Linear

$\left(\frac{\delta z}{\delta x}\right)^2 + \left(\frac{\delta^3 z}{\delta y^3}\right) =$ Non linear

Dependent variable It is having 1st degree & it is not multiplied such as Squares, Cubes, then it is Linear PDE.

$\frac{\delta z}{\delta x} = p$ (small p) $\rightarrow z_x$

$\frac{\delta z}{\delta y} = q \rightarrow z_y$

$\frac{\delta^2 z}{\delta x^2} = r \rightarrow z_{xx}$

$\frac{\delta^2 z}{\delta x \delta y} = \frac{\delta^2 z}{\delta y \delta x} = s \rightarrow z_{xy}, z_{yx}$

$\frac{\delta^2 z}{\delta y^2} = t \rightarrow z_{yy}$

If $z = f(x_1, x_2 \dots x_n)$

$\frac{\delta z}{\delta x_1} = p_1$

$\frac{\delta z}{\delta x_2} = p_2$

$\vdots \quad \vdots$

$\frac{\delta z}{\delta x_n} = p_n$

Classification of 1st Order PDE

Linear | Semi Linear | Quasi Linear | Non-linear

Linear

$$P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$$

Eg:

1. $yx^2p + xy^2q = xyz + x^2y^3$

2. $p+q = z+xy$

General form of 1st order PDE

$$PDE = f(x,y,z,p,q)=0$$

Semi Linear

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

for eg

1. $xyp + x^2yq = x^2y^2z^2$
2. $yp + xq = \dfrac{x^2z^2}{y^2}$

Quasi Linear

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

$$Pp + Qq = R$$

Eg:-

1. $x^2zp + y^2zq = xy$
2. $(x^2-yz)p + (y^2-xz)q = z^2-xy$

Non linear

Eg:

$p^2+q^2=1$

$pq=z$

$x^2p^2+y^2q^2=z^2$

All semilinear is subset of Quasi linear.

Formulation of PDE

Elimination of arbitrary Constants

- No. of arbitrary constants less than Independent Variables
  No. of AC's < no. of IV's
  Usually more than one PDE of order one
- No. of AC's = no. of IV's
  Usually Unique PDE of order one
- No. of AC's > no of IV's
  Usually PDE of Order greater than one

Elimination of Arbitrary functions

No. of AF's = Order of PDE

AF's = Arbitrary functions

$z=ax+y$

$\frac{\delta z}{\delta x}=a$

$\boxed{z=px+y}$

$az+b=a^2x+y \longrightarrow$ w.r.t y

$a\frac{\delta z}{\delta x}=a^2$

$\cancel{a}p=a^{\cancel{2}}$

$\boxed{p=a}$

$a\frac{\delta z}{\delta y}=1$

$aq=1$