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Linear Algebra

IFoS (IFS) Previous Year Questions (PYQ) from 2025 to 2009

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2025

1. If a subspace W of $\mathbb{R}^4$ is generated by the vectors $(3, 8, -3, -5)$, $(1, -2, 5, -3)$ and $(2, 3, 1, -4)$, then find a basis and dimension of W. Extend that basis to get a basis of $\mathbb{R}^4$. **[8 Marks]**

2. Find a row echelon matrix which is row equivalent to **[8 Marks]**

$$A = \begin{bmatrix} 0 & 0 & -2 & 3 & 1 \\ 2 & 4 & 1 & 4 & 3 \\ 1 & 2 & -3 & 1 & 2 \\ 4 & 8 & 2 & 3 & 5 \end{bmatrix}$$

and find the rank of A.

3. For the companion matrix C **[10 Marks]**

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

of an n^{th} degree polynomial

$$\phi(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0,$$

prove that

(i) the characteristic polynomial is $\phi(\lambda)$,

(ii) if λ_i is an eigenvalue of C, then $x_i = [1 \;\; \lambda_i \;\; \lambda_i^2 \;\; \cdots \;\; \lambda_i^{n-1}]^T$ is the associated eigenvector,

(iii) if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigenvalues of C, then

$$V^{-1}CV = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \text{where } V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}.$$

4. Diagonalize the quadratic form **[8 Marks]**

$$5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2.$$

Show that it is positive semi-definite and find a non-zero set of values of x_1, x_2, x_3 which makes the diagonalized form zero.

5. If W is a subspace of a finite dimensional vector space $V(F)$, then prove that W is finite dimensional and $\dim W \leq \dim V$. Also, prove that $\dim W = \dim V$ if and only if $W = V$. **[7 Marks]**

6. For the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + y + z,\ 2y + z,\ 2y + 3z)$, find the eigenvalues and the basis for eigenspace. **[8 Marks]**

7. Prove that the necessary and sufficient condition for a linear transformation $y = Ax$ to preserve lengths is that the matrix A is orthogonal. **[7 Marks]**

2024

8. Let $V = \mathbb{R}^4$. Find a basis and dimension of the subspace **[8 Marks]**

$$W = \{(a, b, c, d) \in V : a = b + c,\ c = b + d\}.$$

9. Describe explicitly a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, which has its range spanned by $(1, 0, -1)$ and $(1, 2, 2)$. **[8 Marks]**

10. Let **[10 Marks]**

$$W_1 = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in \mathbb{C} \right\}, \quad W_2 = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{C} \right\}$$

be two subspaces of the vector space of all 2×2 matrices over the complex field $\mathbb{C}$. Show that

$$\dim\left(\frac{W_1 + W_2}{W_2}\right) = \dim\left(\frac{W_1}{W_1 \cap W_2}\right).$$

11. Reduce the matrix **[15 Marks]**

$$A = \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix}$$

to echelon form and then to row canonical form.

12. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (5x - y + 3z,\ -6x + 4y - 6z,\ -6x + 2y - 4z)$. Find all the eigenvalues and corresponding eigenvectors. **[15 Marks]**

2023

13. Let V be a vector space of the dimension n over a field F. Then show that V is isomorphic to F^n. **[8 Marks]**

14. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by $T(x, y, z) = (x, z, -2y - z)$ and let $f(u) = -u^3 + 2$. Then find $f(T)$. **[8 Marks]**

15. If $S_1 = \{(x, y, z) \mid x + 2y + z = 0\}$ and $S_2 = \{(x, y, z) \mid x + y - z = 0\}$ are subspaces of $\mathbb{R}^3$, then (i) find a basis of $S_1 \cap S_2$, (ii) determine $\dim(S_1 + S_2)$, and (iii) describe $S_1 \cap S_2$ and $S_1 + S_2$ geometrically. **[10 Marks]**

16. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. Prove that T is invertible and find T^{-1}. **[15 Marks]**

17. Show that the matrix **[15 Marks]**

$$A = \begin{bmatrix} 7 & -6 & 6 \\ 2 & 0 & 4 \\ 1 & -2 & 6 \end{bmatrix}$$

is diagonalizable and find a spectral decomposition of the matrix A.

2022

18. Let U and W be subspaces of a vector space V and $x, y \in V$. Then prove that $x + U \subseteq y + W$ if and only if $U \subseteq W$ and $x - y \in W$. **[8 Marks]**

19. Let $v_1 = (1, 1, -1)$, $v_2 = (4, 1, 1)$, $v_3 = (1, -1, 2)$ be a basis of $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that $Tv_1 = (1, 0)$, $Tv_2 = (0, 1)$ and $Tv_3 = (1, 1)$. Describe the linear transformation T. **[8 Marks]**

20. Are the matrices **[10 Marks]**

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

similar? Justify your answer.

21. Reduce the following quadratic form over the real field $\mathbb{R}$ to orthogonal form: **[10 Marks]**

$$q(x, y, z) = x^2 + 5y^2 - 4z^2 + 2xy - 4xz.$$

22. Let V be the complex vector space of 3×3 skew-symmetric matrices with complex entries, i.e. **[15 Marks]**

$$V = \{A \in M_{3\times3}(\mathbb{C}) \mid A^t = -A\}.$$

Let

$$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Define a linear transformation $T : V \to V$ by $T(A) = BA - AB$. Find the eigenvalues and eigenvectors of T.

2021

23. Consider the following quadratic form: **[8 Marks]**

$$q(x, y, z) = 2x^2 + 2y^2 + 6z^2 + 2xy - 6yz - 6zx,$$

where (x, y, z) are the coordinates of the vector X with respect to the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $\mathbb{R}^3$. Find the expression of $q(x, y, z)$ with respect to the basis

$$B = \left\{ \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right\}.$$

Is q positive definite? Justify your answer.

24. Prove that the product of two Hermitian matrices A, B is Hermitian if and only if A and B commute. Give an example of a pair of 3×3 symmetric matrices such that their product is again symmetric (do not consider only diagonal matrices) and also check whether they commute or not. **[8 Marks]**

25. Express the polynomial $f(x) = x^2 + 4x - 3$ over $\mathbb{R}$ as a linear combination of polynomials $e_1 = x^2 - 2x + 5$, $e_2 = 2x^2 - 3x$, $e_3 = x + 3$. Also, show that the set $\{e_1, e_2, e_3\}$ forms a basis of all quadratic polynomials over $\mathbb{R}$. **[10 Marks]**

26. Given the matrix **[15 Marks]**

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix},$$

find a similarity transformation that diagonalises the matrix A.

27. Using the Cayley-Hamilton theorem, find the inverse of the matrix **[15 Marks]**

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix}.$$