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UPSC MATHS OPTIONAL COACHING

Modern Algebra

Previous year Questions

from 2025 to 1992

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2025

1. Let H and K be two subgroups of a group G such that $o(H) > \sqrt{o(G)}$ and $o(K) > \sqrt{o(G)}$. Show that $H \cap K \neq \{e\}$, where e is the identity element. **[10 Marks]**
2. Let $G = \{e, x, x^2, y, yx, yx^2\}$ be a non-abelian group in which $o(x) = 3$ and $o(y) = 2$. Show that $xy = yx^2$, where e is the identity element of G and $o(x), o(y)$ denote the orders of x, y respectively. **[10 Marks]**
3. Show that 3 is an irreducible element in the integral domain $\mathbb{Z}[i]$. **[15 Marks]**
4. Examine whether the mapping $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ defined by $\varphi(f(x)) = f(0)$, for $f(x) \in \mathbb{Z}[x]$, is a homomorphism. Deduce that the ideal (x) is a prime ideal in $\mathbb{Z}[x]$, but not a maximal ideal in $\mathbb{Z}[x]$. **[15 Marks]**

2024

5. Let G be a finite group of order mn, where m and n are prime numbers with $m > n$. Show that G has at most one subgroup of order m. **[10 Marks]**
6. Show that every homomorphic image of an abelian group is abelian, but converse is not necessarily true. **[15 Marks]**
7. Consider the polynomial ring $\mathbb{Z}[x]$ over the ring $\mathbb{Z}$ of integers. Let S be an ideal of $\mathbb{Z}[x]$ generated by x. Show that S is prime but not a maximal ideal of $\mathbb{Z}[x]$. **[15 Marks]**

2023

8. Let G be a group of order 10 and G' be a group of order 6. Examine whether there exists a homomorphism of G onto G'. **[10 Marks]**
9. Express the ideal $4\mathbb{Z} + 6\mathbb{Z}$ as a principal ideal in the integral domain $\mathbb{Z}$. **[10 Marks]**
10. Prove that a non-commutative group of order 2p, where p is an odd prime, must have a subgroup of order p. **[15 Marks]**
11. Prove that $x^2 + 1$ is an irreducible polynomial in $\mathbb{Z}_3[x]$. Further show that the quotient ring $\mathbb{Z}_3[x]/(x^2 + 1)$ is a field of 9 elements. **[15 Marks]**

2022

12. Show that the multiplicative group $G = \{1, -1, i, -i\}$, where $i = \sqrt{-1}$, is isomorphic to the group $G' = (\{0, 1, 2, 3\}, +_4)$. **[10 Marks]**
13. Prove that every homomorphic image of a group G is isomorphic to some quotient group of G. **[15 Marks]**
14. Let R be a field and S be the set of all those polynomials $f(x) \in R[x]$ such that $f(0) = 0 = f(1)$. Prove that S is an ideal of $R[x]$. Is the residue class ring $R[x]/S$ an integral domain? Give justification for your answer. **[15 Marks]**

2021

15. Let $m_1, m_2, \ldots, m_k$ be positive integers and $d > 0$ the greatest common divisor of $m_1, m_2, \ldots, m_k$. Show that there exist integers $x_1, x_2, \ldots, x_k$ such that $d = x_1m_1 + x_2m_2 + \cdots + x_km_k$. **[10 Marks]**
16. Let F be a field and $f(x) \in F[x]$ a polynomial of degree > 0 over F. Show that there is a field F' and embedding $q : F \to F'$ such that the polynomial $f^q \in F'[x]$ has a root in F', where f^q is obtained by replacing each coefficient a of f by $q(a)$. **[15 Marks]**
17. Show that there are infinitely many subgroups of the additive group $\mathbb{Q}$ of rational numbers. **[15 Marks]**

2020

18. Let S_3 and Z_3 be permutation group on 3 symbols and group of residue classes module 3 respectively. Show that there is no homomorphism of S_3 in Z_3 except the trivial homomorphism. **[10 Marks]**

19. Let R be a principal ideal domain. Show that every ideal of a quotient ring of is R principal ideal and R/P is a principal ideal domain for a prime ideal P of R **[10 Marks]**

20. Let G be a finite cyclic group of order n then prove that G has $\phi(n)$ generators where ϕ is Euler's ϕ function. **[15 Marks]**

21. Let R be a finite field of characteristic $p(\geq 0)$ .Show that the mapping $f : R \to R$ defined by $f(a) = a^p, \forall a \in R$ is an isomorphism. **[15 Marks]**

2019

22. Let G be a finite group H and K subgroups of G such that $K \subset H$ Show that $(G:K) = (G:K)(H:K)$ **[10 Marks]**

23. If G and H are finite groups whose orders are relatively prime then prove that there is only one homomorphism from G to H the trivial one. **[10 Marks]**

24. Write down all quotient groups of the group Z_{12} . **[10 Marks]**

25. Let a be an irreducible element of the Euclidean Ring R then prove that $R/(a)$ is a field **[10 Marks]**

2018

26. Let R be an integral domain with unit element. Show that any unit in $R[x]$ is a unit in R **[10 Marks]**

27. Show that the quotient group of $(\mathbb{R},+)$ modulo $\mathbb{Z}$ is isomorphic to the multiplicative group of complex numbers on the unit circle in the complex plane. Here $\mathbb{R}$ is the set of real number and $\mathbb{Z}$ is the set of integers. **[15 Marks]**

28. Find all the proper subgroups of the multiplicative group of the field $(\mathbb{Z}_{13}, +_{13}, \times_{13})$, where $+_{13}$ and $\times_{13}$ represent addition modulo 13 and multiplication modulo 13 respectively. **[20 Marks]**

2017

29. Let G be a group of order n. Show that G is isomorphic to a subgroup of the permutation group S_n . **[10 Marks]**

30. Let F be a field and $F[x]$ denote the ring of polynomial over F in a single variable X . For $f(X), g(X) \in F[X]$ with $g(X) \neq 0$, show that there exist $q(X), r(X) \in F[X]$ such that degree $r(X) <$ degree $g(X)$ and $f(X) = q(X).g(X) + r(X)$. **[20 Marks]**

31. Show that the groups $Z_5 \times Z_7$ and Z_{35} are isomorphic. **[15 Marks]**

2016

32. Let K be a field and $K[X]$ be the ring of polynomials over K in a single variable X for a polynomial $f \in K[X]$ Let (f) denote the ideal in $K[X]$ generated by f. show that (f) is a maximal ideal in $K[X]$ if and only iff is an irreducible polynomial over K . **[10 Marks]**

33. Let p be prime number and Z_p denote the additive group of integers modulo p. show that that every non-zero element Z_p of generates Z_p **[15 Marks]**

34. Let K be an extension of a field F prove that the element of K which are algebraic over F form a subfield of K Further if $F \subset K \subset L$ Fare fields L is algebraic over K and K is algebraic over F then prove that L is algebraic over F . **[20 Marks]**

35. Show that every algebraically closed field is infinite. **[15 Marks]**

2015

36. (i) How many generators are there of the cyclic group G of order 8? Explain. **[5 Marks]**
    (ii) Taking a group $\{e, a, b, c\}$ of order 4, where e is the identity, construct composition tables showing that one is cyclic while the other is not **[5 Marks]**

37. Give an example of a ring having identity but a subring of this having a different identity **[10 Marks]**

38. If R is a ring with unit element 1 and ϕ is a homomorphism of R onto R', prove that $\phi(1)$ is the unit element of R' **[15 Marks]**

39. Do the following sets form integral domains with respect to ordinary addition and multiplication? Is so, state if they are fields: **[5+6+4=15 Marks]**
    (i) The set of numbers of the form $b\sqrt{2}$ with b rational.
    (ii) The set of even integers.
    (iii) The set of positive integers.

2014

40. Let G be the set of all real 2×2 matrices $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$, where $xz \neq 0$. Show that G is group under matrix multiplication. Let N denote the subset $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in R \right\}$. Is N a normal subgroup of G ? Justify your answer. **[10 Marks]**

41. Show that Z_7 is a field. Then find $([5]+[6])^{-1}$ and $(-[4])^{-1}$ in Z_7 **[15 Marks]**

42. Show that the set $\{a + b\omega : \omega^3 = 1\}$, where a and b are real numbers, is a field with respect to usual addition and multiplication. **[15 Marks]**

43. Prove that the set $Q(\sqrt{5}) = \{a + b\sqrt{5} : a, b \in Q\}$ is commutative ring with identity. **[15 Marks]**

2013

44. Show that the set of matrices $S = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} a, b \in R \right\}$ is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Consider the map $f : C \to S$ defined by $f(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Show that f is an isomorphism. (Here R is the set of real numbers and C is the set of complex numbers) **[10 Marks]**

45. Give an example of an infinite group in which every element has finite order **[10 Marks]**

46. What are the orders of the following permutation in S_{10} ? $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix}$ and $(1 \;\; 2 \;\; 3 \;\; 4 \;\; 5)(6 \;\; 7)$ **[10 Marks]**

47. What is the maximal possible order of an element in S_{10} ? Why? Give an example of such an element. How many elements will there be in S_{10} of that order? **[13 Marks]**

48. Let $J = \{a + ib \,/\, a, b \in Z\}$ be the ring of Gaussian integers (subring of C). Which of the following is J : Euclidean domain, principal ideal domain, and unique factorization domain? Justify your answer **[15 Marks]**

49. Let R^C = ring of all real value continuous functions on $[0,\ 1]$, under the operations $(f+g)x = f(x) + g(x),\ (fg)x = f(x)g(x)$. Let $M = \left\{ f \in R^C \,/\, f\left(\frac{1}{2}\right) = 0 \right\}$. Is M a maximal ideal of R? Justify your answer. **[15 Marks]**

2012

50. How many elements of order 2 are there in the group of order 16 generated by a and b such that the order of a is 8, the order of b is 2 and $bab^{-1} = a^{-1}$. **[12 Marks]**

51. How many conjugacy classes does the permutation group S_5 of permutation 5 numbers have? Write down one element in each class (preferably in terms of cycles). **[15 Marks]**

52. Is the ideal generated by 2 and X in the polynomial ring $Z[X]$ of polynomials in a single variable X with coefficients in the ring of integers Z, a principal ideal? Justify your answer **[15 Marks]**

53. Describe the maximal ideals in the ring of Gaussian integers $Z\ [i] = \{a + ib \,/\, a, b \in Z\}$. **[20 Marks]**

2011

54. Show that the set $G = \{f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6\}$ of six transformations on the set of Complex numbers defined by $f_1(z) = z, f_2(z) = 1 - z, f_3(z) = \frac{z}{(1-z)},\ f_4(z) = \frac{1}{z}, f_5(z) = \frac{1}{(1-z)}, f_6(z) = \frac{(z-1)}{z}$ is a non-abelian group of order 6 w.r.t. composition of mappings **[12 Marks]**

55. Prove that a group of Prime order is abelian. **[6 Marks]**

56. How many generators are there of the cyclic group $(G,\ .)$ of order 8? **[6 Marks]**

57. Give an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic **[15 Marks]**

58. Let F be the set of all real valued continuous functions defined on the closed interval $[0,\ 1]$. Prove that $(F,\ +,\ .)$ is a Commutative Ring with unity with respect to addition and multiplication of functions defined point wise as below:

$$\left.\begin{aligned}(f+g)x &= f(x)+g(x)\\ \text{and } (fg)x &= f(x)g(x)\end{aligned}\right\} x\in[0,\ 1] \text{ where } f,\ g\in F$$ **[15 Marks]**

59. Let a and b be elements of a group, with $a^2=e, b^6=e$ and $ab=b^4a$. Find the order of ab, and express its inverse in each of the forms $a^m b^n$ and $b^m a^n$ **[20 Marks]**

2010

60. Let $G=R-\{-1\}$ be the set of all real numbers omitting -1. Define the binary relation $*$ on G by $a*b=a+b+ab$. Show $(G,\ *)$ is a group and it is abelian **[12 Marks]**

61. Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group of order 2 and a cyclic group of order 3. Can you generalize this? Justify. **[12 Marks]**

62. Let $(R^*,\ .)$ be the multiplicative group of non-zero reals and $(GL(n,\ R),\ X)$ be the multiplicative group of $n\times n$ non-singular real matrices. Show that the quotient group $\frac{GL(n,\ R)}{SL(n,\ R)}$ and $(R^*,\ .)$ are isomorphic where $SL(n,\ R)=\{A\in GL(n,\ R)/\det A=1\}$ what is the center of $GL(n,\ R)$ **[15 Marks]**

63. Let $C=\{f: I=[0,\ 1]\to R/f \text{ is continuous }\}$. Show C is a commutative ring with 1 under point wise addition and multiplication. Determine whether C is an integral domain. Explain. **[15 Marks]**

64. Consider the polynomial ring $Q[x]$. Show $p(x)=x^3-2$ is irreducible over Q. Let I be the ideal $Q[x]$ in generated by $p(x)$. Then show that $\frac{Q[x]}{I}$ is a field and that each element of it is of the form $a_0+a_1t+a_2t^2$ with a_0, a_1, a_2 in Q and $t=x+I$ **[15 Marks]**

65. Show that the quotient ring $\frac{Z[i]}{1+3i}$ is isomorphic to the ring $\frac{Z}{10Z}$ where $Z[i]$ denotes the ring of Gaussian integers **[15 Marks]**

2009

66. If R is the set of real numbers and R_+ is the set of positive real numbers, show that R under addition $(R,\ +)$ and R_+ under multiplication $(R_+,\ .)$ are isomorphic. Similarly, if Q is set of rational numbers and Q_+ is the set of positive rational numbers, are $(Q,\ +)$ and $(Q_+,\ .)$ isomorphic? Justify your answer. **[4+8=12 Marks]**

67. Determine the number of homomorphisms from the additive group Z_{15} to the additive group Z_{10} (Z_n is the cyclic group of order n) **[12 Marks]**

68. How many proper, non-zero ideals, does the ring Z_{12} have? Justify your answer. How many ideals does the ring $Z_{12}\oplus Z_{12}$ have? Why? **[2+3+4+6=15Marks]**

69. Show that the alternating group of four letters A_4 has no subgroup of order 6. **[15 Marks]**

70. Show that $Z[X]$ is a unique factorization domain that is not a principal ideal domain (Z is the ring of integers). Is it possible to give an example of principal ideal domain that is not a unique factorization domain? ($Z[X]$ is the ring of polynomials in the variable X with integer.) **[15 Marks]**

71. How many elements does the quotient ring $\frac{Z_5[X]}{X^2+1}$ have? Is it an integral domain? Justify yours answers. **[15 Marks]**

2008

72. Let R_0 be the set of all real numbers except zero. Define a binary operation $*$ on R_0 as $a * b = |a|b$ where $|a|$ denotes absolute value of a. Does $(R_0, *)$ form a group? Examine. **[12 Marks]**

73. Suppose that there is a positive even integer n such that $a^n = a$ for all the elements a of some ring R. Show that $a + a = 0$ for all $a \in R$ and $a + b = 0 \Rightarrow a = b$ for all $a, b \in R$ **[12 Marks]**

74. Let G and $\overline{G}$ be two groups and let $\phi : G \to \overline{G}$ be a homomorphism. For any element $a \in G$
    (i) Prove that $O(\phi(a)) / O(a)$
    (ii) $Ker\ \phi$ is normal subgroup of G. **[15 Marks]**

75. Let R be a ring with unity. If the product of any two non-zero elements is non-zero. Then prove that $ab = 1 \Rightarrow ba = 1$. Whether Z_6 has the above property or not explain. Is Z_6 an integral domain? **[15 Marks]**

76. Prove that every Integral Domain can be embedded in a field. **[15 Marks]**

77. Show that any maximal ideal in the commutative ring $F[x]$ of polynomial over a field F is the principal ideal generated by an irreducible polynomial. **[15 Marks]**

2007

78. If in a group G, $a^5 = e$, e is the identity element of G $aba^{-1} = b^2$ for $a, b \in G$, then find the order of b **[12 Marks]**

79. Let $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in Z$. Show that R is a ring under matrix addition and multiplication $\left\{ A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, a, b \in Z \right\}$. Then show that A is a left ideal of R but not a right ideal of R. **[12 Marks]**

80. (i) Prove that there exists no simple group of order 48. **[15 Marks]**
    (ii) $1 + \sqrt{-3}$ and $Z\left[\sqrt{-3}\right]$ is an irreducible element, but not prime. Justify your answer. **[15 Marks]**

81. Show that in the ring $R = \left\{ a + b\sqrt{-5} / a, b \in Z \right\}$. The element $\alpha = 3$ and $\beta = 1 + 2\sqrt{-5}$ are relatively prime, but αy and βy have no g.c.d in R, where $\gamma = 7\left(1 + 2\sqrt{-5}\right)$ **[30 Marks]**

2006

82. Let S be the set of all real numbers except -1. Define on S by $a * b = a + b + ab$. Is $(S, *)$ a group? Find the solution of the equation $2 * x * 3 = 7$ in S. **[12 Marks]**

83. If G is a group of real numbers under addition and N is the subgroup of G consisting of integers, prove that $\frac{G}{N}$ is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication **[12 Marks]**

84. (i) Let $O(G) = 108$. Show that there exists a normal subgroup or order 27 or 9. **[10 Marks]**
    (ii) Let G be the set of all those ordered pairs (a, b) of real numbers for which $a \neq 0$ and define in G, an operation as follows: $(a, b) \otimes (c, d) = (ac, bc + d)$ Examine whether G is a group w.r.t the operation $\otimes$. If it is a group, is G abelian? **[10 Marks]**

85. Show that $Z\left[\sqrt{2}\right]=\left\{a+b\sqrt{2}:\ a,\ b\in Z\right\}$ is a Euclidean domain. **[30 Marks]**

2005

86. If M and N are normal subgroups of a group G such that $M\cap N=\{e\}$, show that every element of M commutes with every element of N. **[12 Marks]**

87. Show that $(1+i)$ is a prime element in the ring R of Gaussian integers. **[12 Marks]**

88. Let H and K be two subgroups of a finite group G such that $|H|>\sqrt{|G|}$ and $|K|>\sqrt{|G|}$. Prove that $H\cap K\neq\{e\}$. **[15 Marks]**

89. If $f:G\to G^{'}$ is an isomorphism, prove that the order $a\in G$ of is equal to the order of $f(a)$ **[15 Marks]**

90. Prove that any polynomial ring $F[x]$ over a field F is U.F.D **[30 Marks]**

2004

91. If p is prime number of the form $4n+1,\ n$ being a natural number, then show that congruence $x^2\equiv -1(\text{mod}\ p)$ is solvable. **[12 Marks]**

92. Let G be a group such that of all $a,b\in G$ (i) $ab=ba$ (ii) $\left(O(a),O(b)\right)=1$ then show that $O(ab)=O(a)\ O(b)$ **[12 Marks]**

93. Verify that the set E of the four roots of $x^4-1=0$ forms a multiplicative group. Also prove that a transformation $T, T(n)=i^n$ is a homomorphism from I_+ (Group of all integers with addition) onto E under multiplication. **[10 Marks]**

94. Prove that if cancellation law holds for a ring R then $a(\neq 0)\in R$ is not a zero divisor and conversely **[10 Marks]**

95. The residue class ring $\dfrac{Z}{(m)}$ is a field iff m is a prime integer. **[15 Marks]**

96. Define irreducible element and prime element in an integral domain D with units. Prove that every prime element in D is irreducible and converse of this is not (in general) true. **[25 Marks]**

2003

97. If H is a subgroup of a group G such that $x^2\in H$ for every $x\in G$, then prove that H is a normal subgroup of G **[12 Marks]**

98. Show that the ring $Z[i]=\left\{a+ib/a,\ b\in Z, i=\sqrt{-1}\right\}$ of Gaussian integers is a Euclidean domain **[12 Marks]**

99. Let R be the ring of all real-valued continuous functions on the closed interval $[0,\ 1]$. Let $M=\left\{f(x)\in R\Big/ f\left(\dfrac{1}{3}\right)=0\right\}$. Show that M is a maximal ideal of R **[10 Marks]**

100. Let M and N be two ideals of a ring R. Show that $M\cup N$ is an ideal of R if and only if either $M\subseteq N$ or $N\subseteq M$ **[10 Marks]**

101. Show that $Q(\sqrt{3},\ i)$ is a splitting field for $x^5-3x^3+x^2-3$ where Q is the field of rational numbers **[15 Marks]**

102. Prove that x^2+x+4 is irreducible over F the field of integers modulo 11 and prove further that $\frac{F[x]}{(x^2+x+4)}$ is a field having 121 elements. **[15 Marks]**

103. Let R be a unique factorization domain (U.F.D), then prove that $R[x]$ is also U.F.D **[10 Marks]**

2002

104. Show that a group of order 35 is cyclic. **[12 Marks]**

105. Show that polynomial $25x^4+9x^3+3x+3$ is irreducible over the field of rational numbers **[12 Marks]**

106. Show that a group of p^2 is abelian, where p is a prime number. **[10 Marks]**

107. Prove that a group of order 42 has a normal subgroup of order 7. **[10 Marks]**

108. Prove that in the ring $F[x]$ of polynomial over a field F, the ideal $1=|p(x)|$ is maximal if and only if the polynomial $p(x)$ is irreducible over F. **[20 Marks]**

109. Show that every finite integral domain is a field **[10 Marks]**

110. Let F be a field with q elements. Let E be a finite extension of degree n over F. Show that E has q^n elements **[10 Marks]**

2001

111. Let K be a field and G be a finite subgroup of the multiplicative group of non-zero elements of K. Show that G is a cyclic group. **[12 Marks]**

112. Prove that the polynomial $1+x+x^2+x^3+...+x^{p-1}$ where p is prime number is irreducible over the field of rational numbers. **[12 Marks]**

113. Let N be a normal subgroup of a group G. Show that $\frac{G}{N}$ is abelian if and only if for all $x,y\in G,\ xyz^{-1}\in N$ **[20 Marks]**

114. If R is a commutative ring with unit element and M is an ideal of R, then show that maximal ideal of R if and only if $\frac{R}{M}$ is a field **[20 Marks]**

115. Prove that every finite extension of a field is an algebraic extension. Give an example to show that the converse is not true. **[20 Marks]**

2000

116. Let n be a fixed positive integer and let Z_n be the ring of integers modulo n. Let $G=\{\bar{a}\in Z_n a \mid a\neq 0\}$ and a is relatively prime to n. Show that G is a group under multiplication defined in Z_n. Hence, or otherwise, show that $a^{\phi(n)}\equiv a(\operatorname{mod} n)$ for all integers a relatively prime to n where $\phi(n)$ denotes the number of positive integers that are less than n and are relatively prime to n **[20 Marks]**

117. Let M be a subgroup and N a normal subgroup of group G. Show that MN is a subgroup of G and $\frac{MN}{N}$ is isomorphic to $\frac{M}{M\cap N}$. **[20 Marks]**

118. Let F be a finite field. Show that the characteristic of F must be a prime integer p and the number of elements in F must be p^m for some positive integer m. **[20 Marks]**

119. Let F be a field and $F[x]$ denote the set of all polynomials defined over F. If $f(x)$ is an irreducible polynomial in $F[x]$, show that the ideal generated by $f(x)$ in $F[x]$ is maximal and $\frac{F[x]}{f(x)}$ is a field.
[20 Marks]

120. Show that any finite commutative ring with no zero divisors must be a field. **[20 Marks]**

1999

121. If ϕ is a homomorphism of G into $\bar{G}$ with kernel K, then show that K is a normal subgroup of G.
[20 Marks]

122. If p is prime number and $p^{\alpha} / O(G)$, then prove that G has a subgroup of order p^{α}. **[20 Marks]**

123. Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Show that R is a field. **[20 Marks]**

1998

124. Prove that if a group has only four elements then it must be abelian. **[20 Marks]**

125. If H and K are subgroups of a group G then show that HK is a subgroup of G if and if only $HK = KH$.
[20 Marks]

126. Let $(R,+,\ .)$ be a system satisfying all the axioms for a ring with unity with the possible exception of $a+b=b+a$. Prove that $(R,+,\ .)$ is a ring. **[20 Marks]**

127. If p is prime then prove that Z_p is a field. Discuss the case when p is not a prime number. **[20 Marks]**

128. Let D be a principal domain. Show that every element that its neither zero nor a unit in D is a product of irreducible. **[20 Marks]**

1997

129. Show that a necessary and sufficient condition for a subset H of a group G to be a subgroup is $HH^{-1} = H$.
[20 Marks]

130. Show that the order of each subgroup of a finite group is a divisor of the order of the group.
[20 Marks]

131. In a group G, the commutator $(a,b)\ a,b \in G$ is the element $aba^{-1}b^{-1}$ and the smallest subgroup containing all commutators is called the commutator subgroup of G. Show that a quotient group $\frac{G}{H}$ is abelian if and only if H contains the commutator subgroup of G. **[20 Marks]**

132. If $x^2 = x$ for all x in a ring R, show that R is commutative. Give an example to show that the converse is not true. **[20 Marks]**

133. Show that an ideal S of the ring of integers Z is maximal ideal if and only if S is generated by a prime integer. **[20 Marks]**

134. Show that in an integral domain every prime element is irreducible. Give an example to show that the converse is not true. **[20 Marks]**

1996

135. Let R be the set of real numbers and $G = \{(a,b) \mid a,b \in R, a \neq 0\}$. $G \times G \to G$ is defined by $(a,b) * (c,d) = (ac, bc + d)$. Show that $(G,*)$ is a group. Is it abelian? **[20 Marks]**

136. Let f be a homomorphism of a group G onto a group G' with kernel H. For each subgroup K' of G' define K by. Prove that

(i) K' is isomorphic to $\frac{K}{H}$

(ii) $\frac{G}{K}$ is isomorphic to $\frac{G'}{K'}$ **[20 Marks]**

137. Prove that a normal subgroup H of a group G is maximal, if and only if the quotient group $\frac{G}{H}$ is simple. **[20 Marks]**

138. In a ring R, prove that cancellation laws hold. If and only if R has no zero divisors. **[20 Marks]**

139. If S is an ideal of ring R and T any subring of R, then prove that S is an ideal of $S + T = \{s + t \mid s \in S, t \in T\}$. **[20 Marks]**

140. Prove that the polynomial $x^2 + x + 4$ is irreducible over the field of integers modulo 11. **[20 Marks]**

1995

141. Let G be a finite set closed under an associative binary operation such that $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$ for all $a,b,c \in G$. Prove that G is a group. **[20 Marks]**

142. Let G be group of order p^n, where p is a prime number and $n > 0$. Let H be a proper subgroup of G and $N(H) = \{x \in G : x^{-1}hx \in H \ \forall h \in H\}$. Prove that $N(H) \neq H$. **[20 Marks]**

143. Show that a group of order 112 is not simple. **[20 Marks]**

144. Let R be a ring with identity. Suppose there is an element a of R which has more than one right inverse. Prove that a has infinitely many right inverses. **[20 Marks]**

145. Let F be a field and let $p(x)$ be an irreducible polynomial over F. Let $\langle p(x) \rangle$ be the ideal generated by $p(x)$. Prove that $\langle p(x) \rangle$ is a maximal ideal. **[20 Marks]**

146. Let F be a field of characteristic $p \neq 0$. Let $F(x)$ be the polynomial ring. Suppose $f(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$ is an element of $F(x)$. Define $f(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$. If $f(x) = 0$, prove that there exists $g(x) \in F(x)$ such that $f(x) = g(x^p)$. **[20 Marks]**

1994

147. If G is a group such that $(ab)^n = a^nb^n$ for three consecutive integers n for all $a,b \in G$, then prove that G is abelian. **[20 Marks]**

148. Can a group of order 42 be simple? Justify your claim **[20 Marks]**

149. Show that the additive group of integers modulo 4. Is isomorphic to the multiplicative group of the non-zero elements of integers modulo 5. State the two isomorphisms **[20 Marks]**

150. Find all the units of the integral domain of Gaussian integers. **[20 Marks]**

151. Prove or disprove the statement: The polynomial ring $I[x]$ over the ring of integers is a principal ideal ring. **[20 Marks]**

152. If R is an integral domain (not necessarily a unique factorization domain) and F is its field of quotients, then show that any element $f(x)$ in $F(x)$ is of the form $f(x)=\frac{f_0(x)}{a}$ where $f_0(x)\in R[x], a\in R$. **[20 Marks]**

1993

153. If G is a cyclic group of order n and p divides n, then prove that there is a homomorphism of G onto a cyclic group of order p. What is the Kernel of homomorphism? **[20 Marks]**

154. Show that a group of order 56 cannot be simple. **[20 Marks]**

155. Suppose that H, K are normal subgroups of a finite group G with H a normal subgroup of K. If $P=\frac{K}{H}$, $S=\frac{G}{H}$, then prove that the quotient groups $\frac{S}{P}$ and $\frac{G}{K}$ are isomorphic. **[20 Marks]**

156. If Z is the set of integers then show that $Z[\sqrt{-3}]=\left\{a+\sqrt{-3}\ b : a,b\in Z\right\}$ is not a unique factorization domain **[20 Marks]**

157. Construct the addition and multiplication table for $\frac{Z_3[x]}{\langle x^2+1\rangle}$ where Z_3 is the set of integers modulo 3 and $\langle x^2+1\rangle$ is the ideal generated by (x^2+1) in $Z_3[x]$. **[20 Marks]**

158. Let Q be the set of rational number and $Q(2^{1/2},2^{1/3})$ the smallest extension field of Q containing $2^{1/2},2^{1/3}$. Find the basis for $Q(2^{1/2},2^{1/3})$ over Q. **[20 Marks]**

1992

159. If H is a cyclic normal subgroup of a group G, then show that every subgroup of H is normal in G . **[20 Marks]**

160. Show that no group of order 30 is simple. **[20 Marks]**

161. If p is the smallest prime factor of the order of a finite group G, prove that any subgroup of index p is normal. **[20 Marks]**

162. If R is unique factorization domain, then prove that any $f\in R[x]$ is an irreducible element of $R[x]$, if and only if either f is an irreducible element of R or f is an irreducible polynomial in $R[x]$. **[20 Marks]**

163. Prove that x^2+1 and x^2+x+4 are irreducible over F, the field of integers modulo 11. Prove also that $\frac{F[x]}{\langle x^2+1\rangle}$ and $\frac{F[x]}{\langle x^2+x+4\rangle}$ are isomorphic fields each having 121 elements. **[20 Marks]**

164. Find the degree of splitting field $x^5-3x^3+x^2-3$ over Q, the field of rational numbers. **[20 Marks]**