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Real Analysis

Previous year Questions

from 2025 to 1992

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2025

1. Examine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is absolutely or conditionally convergent. **[10 Marks]**

2. Define Cauchy sequence and prove that every convergent sequence of real numbers is a Cauchy sequence. What is the importance of Cauchy condition? **[15 Marks]**

3. Show that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$. **[20 Marks]**

4. Prove that every continuous function is Riemann integrable. **[15 Marks]**

2024

5. Test the convergence of $\int_0^2 \frac{\log x}{\sqrt{2-x}}\, dx$. **[10 Marks]**

6. Using Cauchy general principle of convergence, examine the convergence of the sequence $\{f_n\}$, where $f_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$. **[15 Marks]**

7. Consider the series $\sum_{n=1}^{\infty} U_n(x)$, $0 \le x \le 1$, the sum of whose first n terms is given by $S_n(x) = \frac{1}{2n^2} \log(1 + n^4 x)$, $x \in [0, 1]$. Show that the given series can be differentiated term-by-term, though $\sum_{n=1}^{\infty} U_n'(x)$ does not converge uniformly on $[0, 1]$. **[20 Marks]**

8. Find the upper and lower Riemann integrals for the function f defined on $[0, 1]$ as follows: $f(x) = \sqrt{1 - x^2}$, if x is rational, and $f(x) = 1 - x$, if x is irrational. Hence, show that f is not Riemann integrable on $[0, 1]$. **[15 Marks]**

2023

9. Test the convergence of the series $\sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right] \frac{x^{2n+1}}{2n+1}$, $x > 0$. **[10 Marks]**

10. Using the method of Lagrange's multipliers, find the minimum and maximum distances of the point $P(2, 6, 3)$ from the sphere $x^2 + y^2 + z^2 = 4$. **[15 Marks]**

11. Prove that the oscillation of a real-valued bounded function f defined on $[a, b]$ is the supremum of the set $\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$. **[15 Marks]**

2022

12. Test the convergence of $\int_0^{\infty} \frac{\cos x}{1 + x^2}\, dx$. **[10 Marks]**

13. Let $f(x) = x^2$ on $[0, k]$, $k > 0$. Show that f is Riemann integrable on the closed interval $[0, k]$ and $\int_0^k f\,dx = \frac{k^3}{3}$. **[15 Marks]**

14. Find the maximum and minimum values of $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$, when $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Interpret the result geometrically. **[20 Marks]**

15. Test for convergence or divergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \frac{5^5x^5}{5!} + \cdots$, $x > 0$. **[15 Marks]**

2021

16. Test the uniform convergence of the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$ on $[0, 1]$. **[10 Marks]**

17. If a function f is monotonic in the interval $[a, b]$, then prove that f is Riemann integrable in $[a, b]$. **[10 Marks]**

18. Find the maximum and minimum values of $f(x) = x^3 - 9x^2 + 26x - 24$ for $0 \le x \le 1$. **[15 Marks]**

19. Find the stationary values of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. Interpret the result geometrically. **[20 Marks]**

2020

20. Prove that the sequence (a_n) satisfying the condition $|a_{n+1} - a_n| \le \alpha |a_n - a_{n-1}|$ $0 \le \alpha \le 1$ for all-natural numbers $0 \le \alpha \le 1$ is a Cauchy sequence. **[10 Marks]**

21. Prove that the function $f(x) = \sin x^2$ is not uniformly continuous on the interval $[0, \infty[$ **[15 Marks]**

22. If $u = \tan^{-1}\frac{x^3 + y^3}{x - y}, x \ne y$ then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial u} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$ **[20 Marks]**

23. Show that $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx = \frac{1}{\sqrt{2}}\log_e\left(1 + \sqrt{2}\right)$ **[15 Marks]**

2019

24. Show that the function $f(x, y) = \begin{cases} \frac{x^2 - y^2}{x - y} & (x, y) \ne (1, -1) \quad (1,1) \\ 0 & (x, y) = (1,1) \quad (1, -1) \end{cases}$ is continuous and differentiable at $(1, -1)$ **[10 Marks]**

25. Evaluate $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1 + x^2)}dx, a \ge 0, a \ne 1$ **[10 Marks]**

26. Using differentials, find an approximate value of $f(4.1, 4.9)$ where $f(x, y) = (x^3 + x^2 y)^{\frac{1}{2}}$ **[15 Marks]**

27. Discuss the uniform convergence of $f_n(x) = \dfrac{nx}{1+n^2x^2}, \forall x \in \mathbb{R}(-\infty, \infty) \quad n = 1, 2, 3, \ldots$ **[15 Marks]**

28. Find the maximum value of the $f(x, y, z) = x^2y^2z^2$ subject to the subsidiary condition.
$x^2 + y^2 + z^2 = c^2, (x, y, z \geq 0)$ **[15 Marks]**

29. Discuss the convergence of $\int_1^2 \dfrac{\sqrt{x}}{\ln x} dx$ **[15 Marks]**

2018

30. Prove the inequality: $\dfrac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \dfrac{x}{\sin x} dx < \dfrac{2\pi^2}{9}$ **[10 Marks]**

31. Find the range of $p(>0)$ for which the series: $\dfrac{1}{(1+a)^p} - \dfrac{1}{(2+a)^p} + \dfrac{1}{(3+a)^p} - \ldots, a > 0$
(i) absolutely convergent and (ii) conditionally convergent. **[10 Marks]**

32. Show that if a function f defined on an open interval (a, b) of $\mathbb{R}$ is convex then f is continuous. Show, by example, if the condition of open interval is dropped, then the convex function need not be continuous.
[15 Marks]

33. Suppose $\mathbb{R}$ be the set of all real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a function such that the following equations hold for all $x, y \in \mathbb{R}$:

$(i) f(x + y) = f(x) + f(y)$
$(ii) f(xy) = f(x) f(y)$
Show that $\forall x \in \mathbb{R}$ either $f(x) = 0$, or, $f(x) = x$. **[15 Marks]**

2017

34. Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20},\ n = 1, 2, 3, \ldots$ show that the sequence $x_1, x_2, x_3 ..$ is convergent. **[10 Marks]**

35. Find the Supremum and the Infimum of $\dfrac{x}{\sin x}$ on the interval $\left(0, \dfrac{\pi}{2}\right]$. **[10 Marks]**

36. Let $f(t) = \int_0^t [x]\, dx$ where $[x]$ denote the largest integer less than or equal to x
(i) Determine all the real numbers t at which f is differentiable.
(ii) Determine all the real numbers t at which f is continues but not differentiable. **[15 Marks]**

37. Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100. **[20 Marks]**

2016

38. For that the function $f:(0,\infty)\to R$ given by $f(x)=x^2\sin\frac{1}{x}, 0<x<\infty$ Show that there is a differentiable function $g:R\to R$ that extends f **[10 marks]**

39. Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following:
$x_1=\frac{1}{2}, y_1=1, x_n=\sqrt{x_{n-1}y_{n-1}}, n=2,3,4,\ldots \frac{1}{y_n}=\frac{1}{2}\left(\frac{1}{x_n}+\frac{1}{y_{n-1}}\right), n=2,3,4,\ldots$ and Prove that $x_{n-1}<x_n<y_n<y_{n-1}, n=2,3,4,\ldots$ and deduce that both the sequence converges to the same limit l where $\frac{1}{2}<l<1.$ **[10 marks]**

40. Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$ conditionally convergent (if you use any theorem (s) to show it then you must give a proof of that theorem(s). **[15 marks]**

41. Find the relative maximum minimum values of the function $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$ **[15 marks]**

42. Let $f:R\to R$ be a continuous function such $\lim_{x\to+\infty}f(x)$ and $\lim_{x\to-\infty}f(x)$ exist and are finite. Prove that is uniformly continuous on $\mathbb{R}$ **[15 marks]**

2015

43. Test for convergence $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{n^2+1}\right)$ **[10 Marks]**

44. Is the function $f(x)=\begin{cases}\frac{1}{n}, & \frac{1}{n+1}<x\le\frac{1}{n}\\ 0 & x=0\end{cases}$ Riemann Integrable? If yes, obtain the value of $\int_0^1 f(x)dx$

[15 Marks]

45. Test the series of functions $\sum_{n=1}^{\infty}\frac{nx}{1+n^2x^2}$ for uniform convergence **[15 Marks]**

46. Find the absolute maximum and minimum values of the function $f(x,y)=x^2+3y^2-y$ over the region $x^2+2y^2\le 1$ **[15 Marks]**

2014

47. Test the convergence of the improper integral $\int_1^{\infty}\frac{dx}{x^2\left(1+e^{-x}\right)}$ **[10 Marks]**

48. Integrate $\int_{1}^{0} f(x)\,dx$, ,where $f(x)=\begin{cases} 2x\sin\frac{1}{x}\cos\frac{1}{x}, x\in[0,1] \\ 0 \qquad\qquad x=0 \end{cases}$ **[15 Marks]**

49. Obtain $\frac{\partial^2 f(0,0)}{\partial x\partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y\partial x}$ for the function $f(x,y)=\begin{cases} \frac{xy(3x^2-2y^2)}{x^2+y^2}, (x,y)\neq(0,0) \\ 0 \qquad ,(x,y)\neq(0,0) \end{cases}$

Also, discuss the continuity $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$ of at $(0,0)$ **[15 Marks]**

50. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $xyz=a^3$ by the method of Lagrange multiplies. **[15 Marks]**

2013

51. Let $f(x)=\begin{cases} \frac{x^2}{2}+4 & \text{if } x\geq 0 \\ \frac{-x^2}{2}+2 & \text{if } x<0 \end{cases}$ Is f Riemann integrable in the interval $[-1,2]$? Why? Does there exist a function g such that $g'(x)=f(x)$? Justify your answer. **[10 Marks]**

52. Show that the series $\sum_{1}^{\infty}\frac{(-1)^{n-1}}{n+x^2}$,is uniformly convergent but not absolutely for all real values of x **[13 Marks]**

53. Show that every open subset of R is countable union of disjoint open intervals **[14 Marks]**

54. Let $[x]$ denote the integer part of the real number x, i.e., if $n\leq x<n+1$ where n is an integer, then $[x]=n$. Is the function $f(x)=[x]^2+3$ Riemann integrable in the function in $[-1,2]$? If not, explain why. If it is integrable, compute $\int_{-1}^{2}\left([x]^2+3\right)dx$ **[10 Marks]**

2012

55. Let, $f_n(x)=\begin{cases} 0, & \text{if } x<\frac{1}{n+1} \\ \sin\frac{\pi}{x}, & \text{if } \frac{1}{n+1}\leq x\leq\frac{1}{n} \\ 0, & \text{if } x>\frac{1}{n} \end{cases}$, Show that $f_n(x)$ converges to a continuous function but not uniformly. **[12 Marks]**

56. Show that the series $\sum_{n=1}^{\infty}\left(\frac{\pi}{\pi+1}\right)^n n^6$ is convergent **[12 Marks]**

57. Let $f(x,y)=\begin{cases} \frac{(x+y)^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ 1, & \text{if } (x,y)=(0,0) \end{cases}$ Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ though $f(x,y)$ is not continuous at $(0,0)$. **[15 Marks]**

58. Find the minimum distance of the line given by the planes $3x+4y+5z=7$ and $x-z=9$ and from the origin, by the method of Lagrange's multipliers. **[15 Marks]**

59. Let $f(x)$ be differentiable on $[0,1]$ such that $f(1)=f(0)=0$ and $\int_0^1 f^2(x)\,dx=1$. Prove that

$\int_0^1 xf(x)f'(x)dx=-\frac{1}{2}$ **[15 Marks]**

60. Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify your answer **[20 Marks]**

2011

61. Let $S=(0,1)$ and f be defined by $f(x)=\frac{1}{x}$ where $0<x\le 1$ (in R). Is f uniformly continuous on S ? Justify your answer. **[12 Marks]**

62. Let $f_n(x)=nx(1-x)^n, x\in[0,1]$. Examine the uniform convergence of $\{f_n(x)\}$ on $[0,1]$ **[15 Marks]**

63. Find the shortest distance from the origin $(0,0)$ to the hyperbola $x^2+8xy+7y^2=225$ **[15 Marks]**

64. Show that the series for which the sum of first n terms $f_n(x)=\frac{nx}{1+n^2x^2}, 0\le x\le 1$ cannot be differentiated term-by-term at $x=0$. What happens at $x\neq 0$? **[15 Marks]**

65. Show that if $S(x)=\sum_{n=1}^{\infty}\frac{1}{n^3+n^4x^2}$, then its derivative $S'(x)=-2x\sum_{n=1}^{\infty}\frac{1}{n^2\left(1+nx^2\right)^2}$, for all x **[20 Marks]**

2010

66. Discuss the convergence of the sequence $\{x_n\}$ where $X_n=\frac{\sin\left(\frac{n\pi}{2}\right)}{8}$ **[12 Marks]**

67. Define $\{x_n\}$ by $x_1=5$ and $x_{n+1}=\sqrt{4+x_n}$ for $n>1$ Show that the sequence converges to $\left(\frac{1+\sqrt{17}}{2}\right)$ **[12 Marks]**

68. Define the function $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$. Find $f'(x)$. Is $f'(x)$ continuous at $x=0$? Justify your answer. **[15 Marks]**

69. Consider the series $\sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^2}$. Find the values of x for which it is convergent and also the sum function. Is the converse uniform? Justify your answer. **[15 Marks]**

70. Let $f_n(x) = x^n$ on $-1 < x \le 1$ for $n = 1, 2.....$ Find the limit function. Is the convergence uniform? Justify your answer. **[15 Marks]**

2009

71. State Roll's theorem. Use it to prove that between two roots of $e^x \cos x = 1$ there will be a root o $e^x \sin x = 1$ **[2+10=12 Marks]**

72. Let $f(x) \begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1 \le x < 2 \\ -\frac{|x|}{2}+1 & \text{if } 2 \le x \end{cases}$ What are the points of discontinuity of f, if any? What are the points where f is not differentiable, if any? Justify yours answer. **[12 Marks]**

73. Show that the series $\left(\frac{1}{3}\right)^2 + \left(\frac{1.4}{3.6}\right)^2 + ... + \left(\frac{1.4.7......(3n-2)}{3.6.9............3n}\right)^2 + ...$ converges **[15 Marks]**

74. Show that if $f : [a,b] \to R$ is a continuous function then $f([a,b]) = [c,d]$ form some real numbers c and d, $c \le d$. **[15 Marks]**

75. Show that: $\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{n^2 x^2}{n^4 + x^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1}$ Justify all steps of your answer by quoting the theorems you are using **[15 Marks]**

76. Show that a bounded infinite subset R must have a limit point **[15 Marks]**

2008

77. (i) For $x > 0$, show $\frac{x}{1+x} < \log(1+x) < x$ **[6 Marks]**

(ii) Let $T = \left\{\frac{1}{n}, n \in N\right\} \cup \left\{1 + \frac{3}{2n}, n \in N\right\} \cup \left\{6 - \frac{1}{3n}, n \in N\right\}$. Find derived set T of T. Also find Supremum of T and greatest number of T **[6 Marks]**

78. If $f : R \to R$ is continuous and $f(x+y) = f(x) + f(y)$, for all $x, y \in R$ then show that $f(x) = xf(1)$ for all $x \in R$. **[12 Marks]**

79. Discuss the convergence of the series $\frac{x}{2} + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + ..., x > 0.$ **[15 Marks]**

80. Show that the series $\sum \frac{1}{n(n+1)}$ is equivalent to $\frac{1}{2}\prod_{2}^{\infty}\left(1 + \frac{1}{n^2-1}\right)$ **[15 Marks]**

81. Let f be a continuous function on $[0,1]$. Using first Mean Value theorem on Integration, prove that

$$\lim_{n \to \infty} \int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \frac{\pi}{2}f(0)$$ **[15 Marks]**

82. (i) Prove that the sets $A = [0,1]$, $B = (0,1)$ are equivalent sets. **[6 Marks]**

(ii) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}$, $x \in \left(0, \frac{\pi}{2}\right)$ **[9 Marks]**

2007

83. Show that the function given by $f(x,y)=\begin{cases}\dfrac{xy}{x^2+2y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exists at $(0,0)$ **[12 Marks]**

84. Using Lagrange's mean value theorem, show that $|\cos b-\cos a|\leq|b-a|$ **[12 Marks]**

85. Given a positive real number a and any natural number n, prove that there exists one and only one positive real number ξ such that $\xi^n=a$ **[20 Marks]**

86. Find the volume of the solid in the first octant bounded by the paraboloid $z=36-4x^2-9y^2$ **[20 Marks]**

87. Rearrange the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ to converge to 1 **[20 Marks]**

2006

88. Examine the convergence of $\int_0^1\frac{dx}{x^{1/2}(1-x)^{1/2}}$ **[12 Marks]**

89. Prove that the function f defined by $f(x)=\begin{cases}1 & \text{when } x \text{ is rational}\\ -1 & \text{when } x \text{ is irrational}\end{cases}$ is nowhere continuous. **[12 Marks]**

90. A twice differentiable function f is such that $f(a)=f(b)=0$ and $f(c)>0$ for $a<c<b$. Prove that there is at least one value ξ, $a<\zeta<b$ for which $f''(\zeta)<0$. **[20 Marks]**

91. Show that the function given by $f(x,y)=\begin{cases}\dfrac{x^3+2y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ (i) is continuous at $(0,0)$ (ii) possesses partial derivative $f_x(0,0)$ and $f_y(0,0)$ **[20 Marks]**

92. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **[20 Marks]**

2005

93. If u, v, w are the roots of the equation in λ and $\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1$, evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ **[12 Marks]**

94. Evaluate $\iiint\ln(x+y+z)\,dxdydz$ The integral being extended over all positive values of x,y,z such that $x+y+z\leq 1$ **[12 Marks]**

95. If f' and g' exist for every $x\in[a,b]$ and if $g'(x)$ does not vanish anywhere (a,b), show that there exists c in (a,b) such that $\frac{f(c)-f(a)}{g(b)-g(c)}=\frac{f'(c)}{g'(c)}$ **[30 Marks]**

96. Show that $\int_0^{\infty}e^{-t}t^{n-1}\,dt$ is an improper integral which converges for $n>0$ **[30 Marks]**

2004

97. Show that the function $f(x)$ defined as: $f(x) = \frac{1}{2^n}$, $\frac{1}{2^{n+1}} \le x \le \frac{1}{2^n}$, $n = 0,1,2,.....$ and $f(0) = 0$ is integrable in $[0,1]$, although it has an infinite number of points of discontinuity. Show that $\int_0^1 f(x)dx = \frac{2}{3}$ **[12 Marks]**

98. Show that the function $f(x)$ defined on by: $f(x) = \begin{cases} x & \text{when } x \text{ is irrational} \\ -x & \text{when } x \text{ is rational} \end{cases}$ is continuous only at $x = 0$

[12 Marks]

99. If (x, y, z) be the lengths of perpendiculars drawn from any interior point P of triangle ABC on the sides BC, CA and AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, the sides of the triangle ABC being a, b, c. **[20 Marks]**

100. Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$

[20 Marks]

101. Let $f(x) \ge g(x)$ for every x in $[a,b]$ and f and g are both bounded and Riemann integrable on $[a,b]$. At a point $c \in [a,b]$, let f and g be continuous and $f(c) > g(c)$ then prove that $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$ and hence show that $-\frac{1}{2} < \int_a^b \frac{x^3 \cos 5x}{2 + x^2} dx < \frac{1}{2}$ **[20 Marks]**

2003

102. Let a be a positive real number and $\{x_n\}$ sequence of rational numbers such that $\lim_{n\to\infty} x_n = 0$. Show that $\lim_{n\to\infty} a^{x_n} = 1$ **[12 Marks]**

103. If a continuous function of x satisfies the functional equation $f(x + y) = f(x) + f(y)$ then show that $f(x) = \alpha x$ where α is a constant. **[12 Marks]**

104. Show that the maximum value of $x^2 y^2 z^2$ subject to condition $x^2 + y^2 + z^2 = c^2$ is $\frac{c^{\sim}}{27}$. Interpret the result

[20 Marks]

105. The axes of two equal cylinders intersect at right angles. If a be their radius, then find the volume common to the cylinder by the method of multiple integrals. **[20 Marks]**

106. Show that $\int_0^\infty \frac{dx}{1 + x^2 sin^2 x}$ is divergent **[20 Marks]**

2002

107. Prove that the integral $\int_0^\infty x^{m-1} e^{-x}\,dx$ is convergent if and only if $m > 0$. **[12 Marks]**

108. Find all the positive values of a for which the series $\sum_{n=1}^\infty \frac{(an)^n}{n!}$ converges. **[12 Marks]**

109. Test uniform convergence of the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$, where $p > 0$ **[20 Marks]**

110. Obtain the maxima and minima of $x^2 + y^2 + z^2 - yz - zx - xy$ subject to condition

$x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0$ **[25 Marks]**

111. A solid hemisphere H of radius 'a' has density ρ depending on the distance R from the center of and is given by $\rho = k(2a - R)$ where k is a constant. Find the mass of the hemisphere by the method of multiple integrals **[15 Marks]**

2001

112. Show that $\int_0^{\pi/2} \frac{x^n}{\sin^m x} dx$ exists if and only if $m < n+1$ **[12 Marks]**

113. If $\lim_{n\to\infty} a_n = l$, then prove that $\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots a_n}{n} = l$, **[12 Marks]**

114. A function f is defined in the interval (a,b) as follows

$$f(x) = \begin{cases} \frac{1}{q^2} \text{ when } x = \frac{p}{q} \\ \frac{1}{q^3} \text{ when } x = \sqrt{\frac{p}{q}} \end{cases}$$ where p,q relatively prime integers. $f(x) = 0$ for all other values of x. Is f Riemann integrable? Justify your answer. **[20 Marks]**

115. Show that $U = xy + yz + zx$ has a maximum value when the three variables are connected by the relation $ax + by + cz = 1$ and a,b,c are positive constants satisfying the condition $2(ab + bc + ca) > (a^2 + b^2 + c^2)$ **[25 Marks]**

116. Evaluate $\iiint \left(ax^2 + by^2 + cz^2\right) dxdydz$ taken throughout the region $x^2 + y^2 + z^2 \le R^2$ **[15 Marks]**

2000

117. Given that the terms of a sequence $\{a_n\}$ are such that each $a_k,\ k \le 3$, is the arithmetic mean of its two immediately preceding terms. Show that the sequence converges. Also find the limit of the sequence.

118. Determine the values of x for which the infinite product $\prod_{n=0}^{\infty} \left(1 + \frac{1}{x^{2n}}\right)$ converges absolutely. Find its value whenever it converges. **[12 Marks]**

119. Suppose f is twice differentiable real valued function in $(0,\infty)$ and $M_{0,}M_1$ and M_2 the least upper bounds of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$ respectively in $(0,\infty)$. Prove for each $x > 0,\ h > 0$ that

$f'(x) \frac{1}{2h}\left[f(x+2h) - f(x)\right] - hf'(u)$ for some $u \in (x, x+2h)$. Hence show that $M_1^{\ 2} \le 4M_0M_{2.}$ **[20 Marks]**

120. Evaluate $\iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy\right)$ by transforming into triple integral where S is the closed surface formed by the cylinder $x^2 + y^2 = a^2, 0 \le z \le b$ and the circular disc $x^2 + y^2 \le a^2, z = 0$ and $x^2 + y^2 \le a^2, z = b$ **[20 Marks]**

1999

121. Let A be a subset of the metric space (M,ρ). If (A,ρ) is compact, then show that A is a closed subset of (M,ρ) **[20 Marks]**

122. A sequence $\{S_n\}$ is defined by the recursion formula $S_{n+1}=\sqrt{3S_n}, S_1=1$. Does this sequence converge? If so, find $\lim S_n$ **[20 Marks]**

123. Test for convergence the integral $\int_0^1 x^p\left(\log\frac{1}{x}\right)^q dx$ **[20 Marks]**

124. Find the shortest distance from the origin to the hyperbola $x^2+8xy+7y^2=225,\ z=0$ **[20 Marks]**

125. Show that the double integral $\iint_R \frac{x-y}{(x+y)^3}\,dxdy$ does not exist over $R=[0,1;0,1]$ **[20 Marks]**

126. Verify the Gauss divergence theorem for $\overline{F}=4x\hat{e}_x-2y^2\hat{e}_y+z^2\hat{e}_z$ taken over the region bounded by $x^2+y^2=4,\ z=0$ and $z=3$ where $\hat{e}_x,\hat{e}_y,\hat{e}_z$ are unit vectors along $x-, y-$ and $z-$ directions respectively. **[20 Marks]**

1998

127. Let X be a metric space and $E\subset X$. Show that
   (i) Interior of E is the largest open set contained in E
   (ii) Boundary of E =(closure of E) $\bigcap$ (closure of $X-E$) **[20 Marks]**

128. Let (X,d) and (Y,e) be metric spaces with X compact and $f:X\to Y$ continuous. Show that f is uniformly continuous. **[20 Marks]**

129. Show that the function $f(x,y)=2x^4-3x^2y+y^2$ has $(0,0)$ as the only critical point but the function neither has a minima nor maxima at $(0,0)$ **[20 Marks]**

130. Test the convergence of the integral $\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx,\ \ a\geq 0$ **[20 Marks]**

131. Test the series $\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$ for uniform convergence. **[20 Marks]**

132. Let $f(x)=x$ and $g(x)=x^2$. Does $\int_0^1 fog$ exist? If it exists then find its value **[20 Marks]**

1997

133. Show that a non-empty set P in R^n each of whose points is a limit-point is uncountable. **[20 Marks]**

134. Show that $\iiint_D xyz\ dxdydz=\frac{a^2b^2c^2}{6}$ where domain D is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1$ **[20 Marks]**

135. If $u=\sin^{-1}\left[\left(x^2+y^2\right)^{1/5}\right]$, Prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{2}{25}\tan u(2\tan^2 u-3)$ **[20 Marks]**

1996

136. Let F be the set of all real valued bounded continuous functions defined on the closed interval $[0,1]$. Let d be a mapping of $F \times F$ into R, the set of real numbers, defined by $d(f,g)=\int_0^1 |f(x)-g(x)|\ dx \ \ \forall f,g \in F$. Verify that d is a metric for F **[20 Marks]**

137. Prove that a compact set in a metric space is a closed set. **[20 Marks]**

138. Let $C[a,b]$ denote the set of all functions f on $[a,b]$ which have continuous derivatives at all points of $I=[a,b]$ .For $f,g \in C[a,b]$ define $d(f,g)=|f(a)-g(b)|+\sup\{|f'(x)-g'(x)|, x \in I\}$. Show that the space $(C[a,b],d)$ is a complete. **[20 Marks]**

139. A function f is defined in the interval (a,b) as follows:

$$f(x)=\begin{cases} q^{-2} & \text{when } x=pq^{-1} \\ q^{-3} & \text{when } x=\left(pq^{-1}\right)^{1/2} \end{cases}$$

where p,q are relatively prime integers; $f(x)=0,$ for all other values of x. Is f Riemann integrable? Justify your answer. **[20 Marks]**

140. Test for uniform convergence, the series $\sum_{n=1}^{\infty} \frac{2^n x^{2n-1}}{1+x^{2n}}$ **[20 Marks]**

141. Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin x \sin^{-1}\left(\sin x \sin y\right)\ dxdy$ **[20 Marks]**

1995

142. Let K and F be nonempty disjoint closed subjects of R^2. If K is bounded, show that there exists $\delta>0$ such that $d(x,y)\geq \delta$ for $x \in K$ and $y \in F$ where $d(x,y)$ is the usual distance between x and y. **[20 Marks]**

143. Let f be a continuous real function on R such that f maps open interval into open intervals. Prove that f is monotonic. **[20 Marks]**

144. Let $c_n \geq 0$ for all positive integers n such that is convergent. Suppose $\{S_n\}$ is a sequence of distinct points in (a,b) For $x \in [a,b]$, define $\alpha(x)=\sum c_n \ \{n : x > S_n\}$. Prove that α is an increasing function. If f a continuous real function on $[a,b]$, show that $\int_a^b f d\alpha = \sum c_n f(S_n)$ **[20 Marks]**

145. Suppose f maps an open ball $U \subset R^n$ into R^m and f is differentiable on U. Suppose there exists a real number $M>0$ such that $\|f(x)\| \leq M \ \forall x \in U$. Prove that $|f(b)-f(a)| \leq M|b-a| \ \forall a,b \in U$ **[20 Marks]**

146. Find and classify the extreme values of the function $f(x,y)=x^2+y^2+x+y+xy$ **[20 Marks]**

147. Suppose α is real number not equals to $n\pi$ for any integer n. Prove that

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+2xy\cos\alpha+y^2)}\ dxdy = \frac{\alpha}{2\sin\alpha}$$ **[20 Marks]**

1994

148. Examine the (i) absolute convergence (ii) uniform convergence of the series $(1-x)+x(1-x)+x^2(1-x)+\ldots$ in $[-c,1],\ 0<c<1$ **[20 Marks]**

149. Prove that $S(x)=\sum\frac{1}{n^p+n^q x^2},\ p>1$ is uniformly convergent for all values of x and can be differentiate term by term if $q<3p<2$ **[20 Marks]**

150. Let the function f be defined on $[0,1]$ by the condition $f(x)=2rx$ when $\frac{1}{r+1}<x<\frac{1}{r},\ r>0$ Show that f is Riemann integrable in $[0,1]$ and $\int_0^1 f(x)\ dx=\frac{\pi^2}{6}$ **[20 Marks]**

151. By means of substitution $x+y+z=u, y+z=uv, z=uvw$ evaluate $\iiint (x+y+z)^n xyz\ dxdydz$ taken over the volume bounded by $x=0, y=0, z=0, x+y+z=1$ **[20 Marks]**

1993

152. Examine for Riemann integrability over $[0,2]$ of the function defined in $[0,2]$ by

$$f(x)=\begin{cases} x+x^2, & \text{for rational valuesof } x \\ x^2+x^3, & \text{for irrational values of } x \end{cases}$$

[20 Marks]

153. Prove that $\int_0^\infty \frac{\sin x}{x}\ dx$ converges and conditionally converges. **[20 Marks]**

154. Evaluate $\iiint \frac{dxdydz}{x+y+z+1}$ over the volume bounded by the coordinate planes and the plane $x+y+z=1$

[20 Marks]

1992

155. If we metrize the space of functions continuous on $[a,b]$ by taking $p(x,y)=\sqrt{\int_a^b \left[x(t)-y(t)\right]^2\ dt}$ then show that the resulting metric space is NOT complete **[20 Marks]**

156. Examine $2xyz-4zx-2yz+x^2+y^2+z^2-2x-4y-4z$ for extreme values **[20 Marks]**

157. If $U_n=\frac{1+nx}{ne^{nx}}-\frac{1+(n+1)x}{(n+1)e^{(n+1)x}},\ 0<x<1$ Prove that $\frac{d}{dx}\sum U_n=\sum\frac{d}{dx}U_n$ Is the series uniformly convergent in $(0,1)$? Justify your claim. **[20 Marks]**

158. Find the upper and lower Riemann integral for the function defined in the interval $[0,1]$ as follows

$$\begin{cases} \sqrt{1-x^2} & \text{when } x \text{ is rational} \\ 1-x & \text{when } x \text{ is irrational} \end{cases}$$

and show that is NOT Riemann integrable in $[0,1]$. **[20 Marks]**

159. Discuss the convergence or divergence of $\int_0^{\infty} \frac{x'^{\beta}}{1+x\alpha\sin^2 x}dx,\ \alpha > \beta > 0$ **[20 Marks]**

160. Evaluate $\iint \sqrt{\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\ dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **[20 Marks]**